The polynomial Daugavet property for atomless $L_1(\mu)$-spaces

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Abstract. For any atomless positive measure μ, the space $L_1(\mu)$ has the polynomial Daugavet property, i.e. every weakly compact continuous polynomial $P : L_1(\mu) \longrightarrow L_1(\mu)$ satisfies the Daugavet equation $\|\operatorname{Id} + P\| = 1 + \|P\|$. The same is true for the vector-valued spaces $L_1(\mu, E)$, μ atomless, E arbitrary.

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1. Introduction

Given a Banach space X over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), by B_X we denote the closed unit ball and by S_X the unit sphere of X. We use the notation $\mathbb{T} = S_{\mathbb{K}}$ and $\mathbb{D} = B_{\mathbb{K}}$. Given $k \geqslant 0$, we denote by $\mathcal{P}\left({}^k X; X\right)$ the space of all k-homogeneous polynomials from X to X, and by $\mathcal{P}\left({}^k X\right)$ the space of all k-homogeneous scalar polynomials. We say that $P : X \longrightarrow X$ is a *polynomial* on X, writing $P \in \mathcal{P}(X; X)$, if P is a finite sum of homogeneous polynomials from X to X. Let us recall that $\mathcal{P}(X; X)$ is a normed space if we endow it with the norm $\|P\| = \sup\{\|P(x)\| \ : \ x \in B_X\}$. Therefore, $\mathcal{P}(X; X)$ embeds isometrically into $\ell_\infty(B_X, X)$ (for a Banach space Z and a set Γ, we write $\ell_\infty(\Gamma, Z)$ to denote the Banach space of all bounded functions from Γ to Z endowed with the supremum norm. We will use the notation $\mathcal{P}(X)$ to denote the space of all finite sums of homogeneous scalar polynomials. We always consider $\mathcal{P}(X)$ as a subspace of $\ell_\infty(B_X, \mathbb{K})$.

This paper is devoted to the study of the so-called Daugavet equation for polynomials. In 1963, I. Daugavet [5] showed that every compact linear operator

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T on $C[0,1]$ satisfies $\|\operatorname{Id}+T\| = 1+\|T\|$, a norm equality which is nowadays known as the *Daugavet equation.* A Banach space X is said to have the *Daugavet property* if every weakly compact operator in $L(X)$ satisfies the Daugavet equation. This is the case, among others, of the spaces $C(K,E)$ when the compact space K is perfect (E is any Banach space), $L_1(\mu,E)$ and $L_\infty(\mu,E)$ when the measure μ is atomless, the disk algebra $A(\mathbb{D})$ and the algebra of bounded analytic functions H^∞. We refer the reader to [1, 7, 8] for more information and background.

In 2007 the study of the Daugavet equation was extended to polynomials [3] and, moreover, to bounded functions from the unit ball of a Banach space into the space. Let us recall the relevant definitions. Let X be a real or complex Banach space. A function $\Phi \in \ell_\infty(B_X, X)$ is said to satisfy the *Daugavet equation* if the norm equality

$$\|\operatorname{Id}+\Phi\| = 1 + \|\Phi\| \tag{DE}$$

holds. If this happens for all weakly compact polynomials on X, we say that X has the *polynomial Daugavet property.* The main examples of Banach spaces having the polynomial Daugavet property are: $C_b(\Omega,E)$ when the completely regular space Ω is perfect (E is any Banach space) and its finite-codimensional subspaces, $L_\infty(\mu,E)$ when the measure μ is atomless, and $C_w(K,E)$, $C_{w^*}(K,E^*)$ when the compact space K is perfect. We refer the reader to [3, 4] for more information and background. Let us remark that the polynomial Daugavet property may be characterized in terms of scalar polynomials. We state this result here for the sake of clarity.

Lemma 1.1 ([3, Corollary 2.2])**.** *Let X be a real or complex Banach space. Then, the following are equivalent:*

(i) *X has the polynomial Daugavet property.*
(ii) *For every $p \in \mathcal{P}(X)$ with $\|p\| = 1$, every $x_0 \in S_X$, and every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $y \in B_X$ such that $\operatorname{Re}\omega p(y) > 1-\varepsilon$ and $\|x_0 + \omega y\| > 2-\varepsilon$.*

Let us observe that all the examples above are of "ℓ_∞-type". The aim of this paper is to show that the space $L_1(\mu,E)$ has the polynomial Daugavet property if the measure μ is atomless regardless of the range space E. For the sake of clarity, we will prove in Section 2 the scalar valued case (i.e. when $E = \mathbb{K}$) and then we will explain in Section 3 how to extend these ideas to the vector valued case.

We finish this introduction with an open problem.

Problem 1.2. *Does the Daugavet property imply the polynomial Daugavet property?*

2. Scalar-valued case

Let (Ω,Σ,μ) be a measure space. $L_1(\mu)$ denotes the space of all measurable scalar valued functions whose moduli have finite integral. The set of all simple (i.e. finite-valued integrable) functions will be denoted by $\mathcal{S}(\mu)$. Two observations are pertinent. First, every element in $\mathcal{S}(\mu)$ has finite support. Second, $\mathcal{S}(\mu)$ is dense in $L_1(\mu)$.

Definition 2.1. Let (Ω, Σ, μ) be a measure space. Let $z \in \mathcal{S}(\mu)$. We say that a simple function $x \in \mathcal{S}(\mu)$ is a *splitting* of z (or that x *splits* z) provided $x^2 = z^2$ and $\displaystyle\int_{z^{-1}(\{a\})} x\, d\mu = 0$ for each $a \in \mathbb{K}$.

Remark 2.2. Suppose that $z = \sum_{j=1}^{m} a_j \mathbf{1}_{A_j} \in \mathcal{S}(\mu)$ with some $0 \neq a_j \in \mathbb{K}$ and pairwise disjoint elements $A_j \in \Sigma$, and that $x \in \mathcal{S}(\mu)$ splits z. We set

$$A_j^+ = \big\{\omega \in A_j :\ x(\omega) = a_j\big\} \quad \text{and} \quad A_j^- = \big\{\omega \in A_j :\ x(\omega) = -a_j\big\}.$$

With this notation, the fact that x splits z exactly means that $A_j = A_j^+ \cup A_j^-$ and $\mu(A_j^+) = \mu(A_j^-)$ for each $j = 1, \ldots, m$.

The following result gives two easy properties of splitting functions which we will need in the sequel.

Proposition 2.3. *Let (Ω, Σ, μ) be a measure space. If $x \in \mathcal{S}(\mu)$ splits $z \in \mathcal{S}(\mu)$, then*

$$\|x + z\| = \|z\| \quad \text{and} \quad \mu\big(\mathrm{Supp}(x + z)\big) = \frac{1}{2}\,\mu(\mathrm{Supp}\, z).$$

Proof. We use the notation given in Remark 2.2. Since $(x+z)\mathbf{1}_{A_j} = 2z\mathbf{1}_{A_j^+}$, one has that

$$\|x+z\| = \sum_{j=1}^{m} \int_{A_j} |x+z| d\mu = \sum_{j=1}^{m} \int_{A_j^+} 2|a_j| d\mu = \sum_{j=1}^{m} |a_j| 2\mu(A_j^+) = \sum_{j=1}^{m} |a_j| \mu(A_j) = \|z\|$$

and

$$\mu\big(\mathrm{supp}\,(x+z)\big) = \mu\Big(\bigcup_{j=1}^{m} A_j^+\Big) = \sum_{j=1}^{m} \mu(A_j^+) = \frac{1}{2}\sum_{j=1}^{m} \mu(A_j) = \frac{1}{2}\mu(\mathrm{supp}\, z). \qquad \square$$

The following technical lemma, which may have independent interest, will be the key to get our results.

Lemma 2.4. *Let (Ω, Σ, μ) be an atomless measure space. For any $z \in \mathcal{S}(\mu)$ there exists a weakly null sequence (x_n) in $\mathcal{S}(\mu)$ such that each x_n splits z.*

Proof. Suppose that $z = \sum_{j=1}^{m} a_j \mathbf{1}_{A_j}$ with some $a_j \in \mathbb{K}$ and pairwise disjoint elements $A_j \in \Sigma$. Using that μ is atomless, for each $j = 1, \ldots, m$ fixed, one can recursively construct a collection of sets $A_{j,n,k} \in \Sigma$ for $n \in \mathbb{N}$ and $k = 1, \ldots, 2^n$ with the following properties:

$$A_{j,0,1} = A_j, \quad A_{j,n,k} = A_{j,n+1,2k-1} \cup A_{j,n+1,2k} \quad \text{and} \quad \mu(A_{j,n,k}) = 2^{-n}\mu(A_j)$$

for any $n \in \mathbb{N}$, $k = 1, \ldots, 2^n$. Then the so-called generalized Rademacher system

$$r_{j,n} = \sum_{k=1}^{2^n} (-1)^k \mathbf{1}_{A_{j,n,k}} \qquad (n \in \mathbb{N})$$

is a weakly null sequence in $L_1(\mu)$ (see [1, p. 497], for instance). It clearly follows that the sequence of simple functions (x_n) defined by $x_n = \sum_{j=1}^{m} a_j\, r_{j,n}$ for every $n \in \mathbb{N}$, is weakly null and it has been constructed in such a way that each x_n is a splitting of z. □

We may now apply the above result to obtain the following lemma.

Lemma 2.5. *Let (Ω, Σ, μ) be an atomless measure space and let $\varphi \in \ell_\infty(B_{L_1(\mu)}, \mathbb{K})$ be weakly sequentially continuous. Given $\varepsilon > 0$ and $\delta > 0$, there is $y \in \mathcal{S}(\mu)$ satisfying*

$$\|y\| = 1, \quad \mu(\operatorname{Supp} y) < \delta \quad \textit{and} \quad \bigl|\varphi(y)\bigr| > \|\varphi\| - \varepsilon.$$

Proof. We start by picking $y_0 \in \mathcal{S}(\mu)$ and $n \in \mathbb{N}$ such that

$$|\varphi(y_0)| > \|\varphi\| - \varepsilon \quad \text{and} \quad \frac{1}{2^n}\mu\bigl(\operatorname{Supp}(y_0)\bigr) < \delta.$$

Now, we use Lemma 2.4 to obtain a weakly null sequence (x_k) in $\mathcal{S}(\mu)$ such that each x_k splits y_0. Therefore, $\lim_k \varphi(x_k + y_0) = \varphi(y_0)$ and hence we may and do choose $k \in \mathbb{N}$ so that $|\varphi(x_k + y_0)| > \|\varphi\| - \varepsilon$. By Proposition 2.3, $y_1 = x_k + y_0$ satisfies $\|y_1\| = 1$ and $\mu(\operatorname{Supp} y_1) = \frac{1}{2}\mu(\operatorname{Supp} y_0)$. Finally, we are done by just repeating the argument n times. □

We are now ready to prove the main result of the section.

Theorem 2.6. *Let (Ω, Σ, μ) be an atomless measure space. Then the space $L_1(\mu)$ has the polynomial Daugavet property.*

Proof. Let us start observing that since $L_1(\mu)$ has the Dunford-Pettis property (see [2, Theorem 5.4.5] for instance), every scalar polynomial on $L_1(\mu)$ is weakly sequentially continuous (see [6, Proposition 2.34] for instance).

We will show that $L_1(\mu)$ has the polynomial Daugavet property by using the characterization given in Lemma 1.1. To do so, we fix $p \in \mathcal{P}(L_1(\mu))$ with $\|p\| = 1$, $x_0 \in S_X$ and $\varepsilon > 0$. We pick $\delta > 0$ so that $\int_A |x_0|d\mu < \varepsilon/2$ for each $A \in \Sigma$ with $\mu(A) \leqslant \delta$. As p is weakly sequentially continuous, we can use Lemma 2.5 to choose $y \in \mathcal{S}(\mu)$ with $\|y\| = 1$, $\mu(\operatorname{Supp} y) \leqslant \delta$ and $\bigl|p(y)\bigr| > 1 - \varepsilon$. If we pick $\omega \in \mathbb{T}$ such that $\operatorname{Re} \omega\, p(y) = \bigl|p(y)\bigr|$, we get $\operatorname{Re} \omega\, p(y) > 1 - \varepsilon$ and

$$\begin{aligned}\|x_0 + \omega\, y\| &= \int_{\Omega \setminus \operatorname{Supp} y} |x_0|\, d\mu + \int_{\operatorname{Supp} y} |x_0 + \omega y|\, d\mu \\ &\geqslant \|x_0\| - \int_{\operatorname{Supp} y} |x_0|\, d\mu + \|y\| - \int_{\operatorname{Supp} y} |x_0|\, d\mu > 2 - \varepsilon.\end{aligned}$$

The result now follows from Lemma 1.1. □

Let us observe that the only property on scalar polynomials on $L_1(\mu)$ we have used in the previous proof is that they are weakly sequentially continuous. Actually, we get the following more general result.

Proposition 2.7. *Let (Ω, Σ, μ) be an atomless measure space, let $\varphi \in \ell_\infty(B_{L_1(\mu)}, \mathbb{K})$ be weakly sequentially continuous with $\|\varphi\| = 1$, and let $x_0 \in S_{L_1(\mu)}$. Then for every $\varepsilon > 0$ there exist $\omega \in \mathbb{T}$ and $y \in L_1(\mu)$, $\|y\| \leqslant 1$, such that*

$$\operatorname{Re} \omega\, \varphi(y) > 1 - \varepsilon \qquad \text{and} \qquad \|x_0 + \omega\, y\| > 2 - \varepsilon.$$

Equivalently, every weakly sequentially continuous bounded function from $B_{L_1(\mu)}$ to $L_1(\mu)$ with relatively weakly compact range satisfies (DE).

Proof. For the first part, just follow the proof of Theorem 2.6. The equivalent reformulation follows from the first part of the proposition using [3, Theorem 1.1] (this is the analogous result to Lemma 1.1 for other subspaces of $\ell_\infty(B_X, X)$). ☐

3. Vector-valued case

Let (Ω, Σ, μ) be a measure space and let E be a Banach space. $L_1(\mu, E)$ denotes the Banach space of all Böchner-integrable functions from Ω to E, endowed with the norm

$$\|f\| = \int_\Omega \|f(t)\|\, d\mu \qquad \big(f \in L_1(\mu, E)\big).$$

All the results given in the previous section can be stated and proved in the setting of $L_1(\mu, E)$ instead of $L_1(\mu)$. The proof of most of them requires just minor modifications with respect to the same given there, but we need to be careful to get the analogous to Theorem 2.6 since, in general, the space $L_1(\mu, E)$ does not have the Dunford-Pettis property.

We need some notation. If F is a closed subspace of E, we will consider $L_1(\mu, F)$ as a subspace of $L_1(\mu, E)$ in the natural way. The set of all simple (i.e. finite-valued integrable) functions $x \in L_1(\mu, E)$ is denoted by $\mathcal{S}(\mu, E)$. We will use in the sequel the following facts: every element of $\mathcal{S}(\mu, E)$ has finite support and $\mathcal{S}(\mu, E)$ is dense in $L_1(\mu, E)$.

Definition 3.1. Let (Ω, Σ, μ) be a measure space, let E be a Banach space and let F be a subspace of E. Let $z \in \mathcal{S}(\mu, F)$. We say that a simple function $x \in \mathcal{S}(\mu, F)$ is a *splitting* of z (or that x *splits* z) provided $x(\omega) \in \{-z(\omega), z(\omega)\}$ a.e. and $\displaystyle\int_{z^{-1}(\{a\})} x\, d\mu = 0$ for each $a \in F$.

Next result summarizes the analogous results to those given in Section 2 for weakly sequentially continuous bounded functions. We omit the proofs which are exactly the same as the ones given there.

Proposition 3.2. *Let (Ω, Σ, μ) be a measure space and let E be a Banach space.*

(a) *If $x \in \mathcal{S}(\mu, E)$ splits $z \in \mathcal{S}(\mu, E)$, then*

$$\|x + z\| = \|z\| \quad \text{and} \quad \mu\big(\operatorname{Supp}(x + z)\big) = \frac{1}{2}\, \mu(\operatorname{Supp} z).$$

(b) *For any $z \in \mathcal{S}(\mu, E)$ there is a weakly null sequence (x_n) in $\mathcal{S}(\mu, E)$ such that each x_n splits z.*

(c) *Let $\varphi \in \ell_\infty(B_{L_1(\mu,E)},\mathbb{K})$ be weakly sequentially continuous. Given $\varepsilon > 0$ and $\delta > 0$, there is $y \in \mathcal{S}(\mu,E)$ satisfying*

$$\|y\| = 1, \quad \mu(\operatorname{Supp} y) < \delta \quad \textit{and} \quad \big|\varphi(y)\big| > \|\varphi\| - \varepsilon.$$

(d) *Let $\varphi : B_{L_1(\mu,E)} \longrightarrow \mathbb{K}$ be a weakly sequentially continuous bounded function with $\|\varphi\| = 1$, and let $x_0 \in S_{L_1(\mu,E)}$. Then for every $\varepsilon > 0$ there exist $\omega \in \mathbb{T}$ and $y \in L_1(\mu,E)$, $\|y\| \leqslant 1$, such that*

$$\operatorname{Re} \omega\, \varphi(y) > 1 - \varepsilon \qquad \textit{and} \qquad \|x_0 + \omega\, y\| > 2 - \varepsilon.$$

Equivalently, every $\Phi \in \ell_\infty(B_{L_1(\mu,E)}, L_1(\mu,E))$ weakly sequentially continuous with relatively weakly compact range satisfies (DE).

The proof of the result for polynomials (i.e. of Theorem 2.6) is not a simple adaptation of the one given in the scalar valued case since $L_1(\mu,E)$ does not have the Dunford-Pettis property in general. Even so, the result can be proved using directly item (d) of the above proposition.

Theorem 3.3. *Let (Ω,Σ,μ) be a measure space and let E be a Banach space. Then $L_1(\mu,E)$ has the polynomial Daugavet property.*

Proof. We fix $p \in \mathcal{P}\left(L_1(\mu,E)\right)$ with $\|p\| = 1$, $x \in L_1(\mu,E)$ with $\|x\| = 1$ and $\varepsilon > 0$. As $\mathcal{S}(\mu,E)$ is dense in $L_1(\mu,E)$, we can find $x_0, x_1 \in \mathcal{S}(\mu,E)$ with $\|x_0\| = 1$ and $\|x_1\| = 1$ such that

$$\|x - x_0\| < \varepsilon/2 \quad \text{and} \quad |p(x_1)| > 1 - \varepsilon/2.$$

Let F be a finite-dimensional subspace of E such that the ranges of x_0 and x_1 belong to F (i.e. $x_0, x_1 \in L_1(\mu,F)$) and consider $\widetilde{p} \in \mathcal{P}\left(L_1(\mu,F)\right)$ as the restriction of p to $L_1(\mu,F)$. Then, $\|\widetilde{p}\| \geqslant |p(x_1)| > 1 - \varepsilon/2$. On the other hand, as F is finite-dimensional, the space $L_1(\mu,F)$ has the Dunford-Pettis property (indeed, as F is finite-dimensional, $L_1(\mu,F)$ is isomorphic to an $L_1(\nu)$ space) and so every scalar polynomial on $L_1(\mu,F)$ is weakly sequentially continuous (see [6, Proposition 2.34] for instance). Now, we may apply item (d) of Proposition 3.2 to $\varphi = \widetilde{p}/\|\widetilde{p}\|$ (which is sequentially continuous on $L_1(\mu,F)$), $x_0 \in S_{L_1(\mu,F)}$ and $\varepsilon/2$ to get $y \in L_1(\mu,F) \subset L_1(\mu,E)$ with $\|y\| \leqslant 1$ and $\omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega\, \varphi(y) > 1 - \varepsilon/2 \qquad \text{and} \qquad \|x_0 + \omega\, y\| > 2 - \varepsilon/2.$$

It follows that

$$\operatorname{Re} \omega\, p(y) = \operatorname{Re} \omega\, \widetilde{p}(y) > (1 - \varepsilon/2) \operatorname{Re} \omega\, \varphi(y) > 1 - \varepsilon,$$

and

$$\|x + \omega\, y\| \geqslant \|x_0 + \omega\, y\| - \|x - x_0\| > 2 - \varepsilon/2 - \varepsilon/2 = 2 - \varepsilon.$$

Finally, $L_1(\mu,E)$ has the polynomial Daugavet property by Lemma 1.1. $\square$

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