ON THE NUMERICAL INDEX OF REAL $L_p(\mu)$-SPACES

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ABSTRACT. We give a lower bound for the numerical index of the real space $L_p(\mu)$ showing, in particular, that it is non-zero for $p \neq 2$. In other words, it is shown that for every bounded linear operator T on the real space $L_p(\mu)$, one has

$$\sup\left\{\left|\int |x|^{p-1}\operatorname{sign}(x)\,Tx\,d\mu\right| \ : \ x \in L_p(\mu),\ \|x\| = 1\right\} \geqslant \frac{M_p}{12\,\mathrm{e}}\,\|T\|$$

where $M_p = \max\limits_{t \in [0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p} > 0$ for every $p \neq 2$. It is also shown that for every bounded linear operator T on the real space $L_p(\mu)$, one has

$$\sup\left\{\int |x|^{p-1}|Tx|\,d\mu \ : \ x \in L_p(\mu),\ \|x\| = 1\right\} \geqslant \frac{1}{2\,\mathrm{e}}\,\|T\|.$$

1. INTRODUCTION

The numerical index of a Banach space is a constant introduced by G. Lumer in 1968 (see [3]) which relates the norm and the numerical radius of (bounded linear) operators on the space. Let us start by recalling the relevant definitions. Given a Banach space X, we will write X^* for its topological dual and $\mathcal{L}(X)$ for the Banach algebra of all (bounded linear) operators on X. For an operator $T \in \mathcal{L}(X)$, its *numerical radius* is defined as

$$v(T) := \sup\{|x^*(Tx)| \ : \ x^* \in X^*,\ x \in X,\ \|x^*\| = \|x\| = x^*(x) = 1\},$$

and it is clear that v is a seminorm on $\mathcal{L}(X)$ smaller than the operator norm. The *numerical index* of X is the constant given by

$$n(X) := \inf\{v(T) \ : \ T \in \mathcal{L}(X),\ \|T\| = 1\}$$

or, equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ such that $k\,\|T\| \leqslant v(T)$ for every $T \in \mathcal{L}(X)$. Classical references here are the aforementioned paper [3] and the monographs by F. Bonsall and J. Duncan [1, 2] from the seventies. The reader will find the state-of-the-art on the subject in the recent survey paper [7] and references therein. We refer to all these references for background.

Let us comment on some results regarding the numerical index which will be relevant in the sequel. First, it is clear that $0 \leqslant n(X) \leqslant 1$ for every Banach space X, and $n(X) > 0$ means that the numerical radius and the operator norm are equivalent on $\mathcal{L}(X)$. In the real case, all values in $[0, 1]$ are possible for the numerical index. In the complex case one has $1/\mathrm{e} \leqslant n(X) \leqslant 1$ and all of these values are possible. Let us also mention that $n(X^*) \leqslant n(X)$, and that the equality does not always hold. Anyhow, when X is a reflexive space, one clearly gets $n(X) = n(X^*)$. Second, there are some classical Banach spaces for which the numerical index has been calculated. For instance, the numerical index of $L_1(\mu)$ is 1, and this property is shared by any of its isometric preduals. In particular, $n\big(C(K)\big) = 1$ for every compact K and $n(Y) = 1$ for every finite-codimensional subspace Y of $C[0, 1]$. If H is a

Hilbert space of dimension greater than one then $n(H) = 0$ in the real case and $n(H) = 1/2$ in the complex case.

Let (Ω, Σ, μ) be a measure space and $1 < p < \infty$. We write $L_p(\mu)$ for the real or complex Banach space of measurable scalar functions x defined on Ω such that

$$\|x\|_p := \left(\int_\Omega |x|^p \, d\mu \right)^{\frac{1}{p}} < \infty.$$

We use the notation ℓ_p^m for the m-dimensional L_p-space. For $A \in \Sigma$, χ_A denotes the characteristic function of the set A. We write $q = p/(p-1)$ for the conjugate exponent to p and

$$M_p := \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = \max_{t \geqslant 1} \frac{|t^{p-1} - t|}{1 + t^p} \, ,$$

(which is the numerical radius of the operator $T(x, y) = (-y, x)$ defined on the real space ℓ_p^2, see [8, Lemma 2] for instance).

The problem of computing the numerical index of the L_p-spaces was posed for the first time in the seminal paper [3, p. 488]. There it is proved that $\left\{ n(\ell_p^2) \ : \ 1 < p < \infty \right\} = [0, 1[$ in the real case, even though the exact computation of $n(\ell_p^2)$ is not achieved for $p \neq 2$ (even now!). Recently, some results have been obtained on the numerical index of the L_p-spaces [4, 5, 6, 8, 9].

(a) The sequence $\left(n(\ell_p^m) \right)_{m \in \mathbb{N}}$ is decreasing.
(b) $n\left(L_p(\mu) \right) = \inf\{ n(\ell_p^m) : m \in \mathbb{N} \}$ for every measure μ such that $\dim\left(L_p(\mu) \right) = \infty$.
(c) In the real case, $\max\left\{ \dfrac{1}{2^{1/p}}, \ \dfrac{1}{2^{1/q}} \right\} M_p \leqslant n(\ell_p^2) \leqslant M_p$.
(d) In the real case, $n(\ell_p^m) > 0$ for $p \neq 2$ and $m \in \mathbb{N}$.

The aim of this paper is to give a lower estimation for the numerical index of the *real* L_p-spaces. Concretely, it is proved that

$$(1) \qquad\qquad\qquad\qquad\qquad n\left(L_p(\mu) \right) \geqslant \frac{M_p}{12\,\mathrm{e}} \, .$$

As $M_p > 0$ for $p \neq 2$, this extends item (d) for infinite-dimensional real L_p-spaces, meaning that the numerical radius and the operator norm are equivalent on $\mathcal{L}\left(L_p(\mu) \right)$ for every $p \neq 2$ and every positive measure μ. This answers in the positive a question raised by C. Finet and D. Li (see [5, 6]) also posed in [7, Problem 1].

The key idea to get this result is to define a new seminorm on $\mathcal{L}\left(L_p(\mu) \right)$ which is in between the numerical radius and the operator norm, and to get constants of equivalence between these three seminorms. Let us give the corresponding definitions.

For any $x \in L_p(\mu)$, we denote

$$x^{\#} = \begin{cases} |x|^{p-1} \operatorname{sign}(x) & \text{in the real case,} \\ |x|^{p-1} \operatorname{sign}(\overline{x}) & \text{in the complex case,} \end{cases}$$

which is the unique element in $L_q(\mu)$ such that

$$\|x\|_p^p = \|x^{\#}\|_q^q \qquad \text{and} \qquad \int_\Omega x \, x^{\#} \, d\mu = \|x\|_p \, \|x^{\#}\|_q = \|x\|_p^p.$$

With this notation, for $T \in \mathcal{L}\left(L_p(\mu) \right)$ one has

$$v(T) = \sup\left\{ \left| \int_\Omega x^{\#} T x \, d\mu \right| \ : \ x \in L_p(\mu), \ \|x\|_p = 1 \right\}.$$

Here is our new definition. Given an operator $T \in \mathcal{L}\big(L_p(\mu)\big)$, the *absolute numerical radius* of T is given by

$$|v|(T) := \sup\left\{ \int_\Omega |x^\# Tx|\, d\mu \ : \ x \in L_p(\mu),\ \|x\|_p = 1 \right\}$$
$$= \sup\left\{ \int_\Omega |x|^{p-1}|Tx|\, d\mu \ : \ x \in L_p(\mu),\ \|x\|_p = 1 \right\}$$

Obviously,

$$v(T) \leqslant |v|(T) \leqslant \|T\| \qquad \big(T \in \mathcal{L}\big(L_p(\mu)\big)\big).$$

Given an operator T on the *real* space $L_p(\mu)$, we will show that

$$v(T) \geqslant \frac{M_p}{6}\,|v|(T) \qquad \text{and} \qquad |v|(T) \geqslant \frac{n\big(L_p^{\mathbb{C}}(\mu)\big)}{2}\,\|T\|,$$

where $n\big(L_p^{\mathbb{C}}(\mu)\big)$ is the numerical index of the *complex* space $L_p(\mu)$. Since $n\big(L_p^{\mathbb{C}}(\mu)\big) \geqslant 1/\mathrm{e}$ (as for any complex space, see [1, Theorem 4.1]), the above two inequalities together give, in particular, the inequality (1).

2. The results

We start proving that the numerical radius is bounded from below by some multiple of the absolute numerical radius.

Theorem 1. *Let $1 < p < \infty$ and let μ be a positive meassure. Then, every bounded linear operator T on the real space $L_p(\mu)$ satisfies*

$$v(T) \geqslant \frac{M_p}{6}\,|v|(T),$$

where $M_p = \max\limits_{t \geqslant 1} \dfrac{|t^{p-1} - t|}{1 + t^p}$.

Proof. Since $|v|$ is a seminorm, we may and do assume that $\|T\| = 1$. Suppose that $|v|(T) > 0$ (otherwise there is nothing to prove), fix any $0 < \varepsilon < |v|(T)$ and choose $x \in L_p(\mu)$ with $\|x\| = 1$ such that

$$\int_\Omega |x^\# Tx|\, d\mu \geqslant |v|(T) - \varepsilon \stackrel{def}{=} 2\beta_0 > 0.$$

Now, set $A = \{t \in \Omega \ : \ x^\#(t)[Tx](t) \geqslant 0\}$ and $B = \Omega \setminus A$. Then

$$\int_A x^\# Tx\, d\mu - \int_B x^\# Tx\, d\mu = \int_\Omega |x^\# Tx|\, d\mu \geqslant 2\beta_0$$

and so at least one of the summands above is greater than or equal to β_0. Without loss of generality, we assume that

$$\beta \stackrel{def}{=} \int_A x^\# Tx\, d\mu \geqslant \beta_0$$

(otherwise we consider $-T$ instead of T). Remark that

$$(2) \qquad \left|\int_\Omega x^\# Tx\, d\mu\right| \leqslant v(T) \qquad \text{and} \qquad \left|\int_B x^\# T(x\chi_B)\, d\mu\right| \leqslant \big\|(x\chi_B)^\#\big\|_q \|x\chi_B\|_p\, v(T) \leqslant v(T).$$

Now, put $y_\lambda = x + \lambda x\chi_B$ for each $\lambda \in [-1, \infty)$. Observe that

$$(3) \qquad \|y_\lambda^\#\|_q \|y_\lambda\|_p = \|y_\lambda\|_p^p = \int_A |x|^p\, d\mu + (1+\lambda)^p \int_B |x|^p\, d\mu \leqslant \max\{1, (1+\lambda)^p\},$$

which obviously implies that

$$(4) \qquad \left|\int_\Omega y_\lambda^\# Ty_\lambda\, d\mu\right| \leqslant v(T)\,\big\|y_\lambda^\#\big\|_q \|y_\lambda\|_p \leqslant v(T)\max\{1, (1+\lambda)^p\}.$$

On the other hand, using that $y_\lambda^\# = x^\# \chi_A + (1+\lambda)^{p-1} x^\# \chi_B$ and (2), we deduce that

$$
\left| \int_\Omega y_\lambda^\# T y_\lambda \, d\mu \right| = \left| \beta + \lambda \int_A x^\# T(x\chi_B) \, d\mu + (1+\lambda)^{p-1} \int_B x^\# Tx \, d\mu + \lambda(1+\lambda)^{p-1} \int_B x^\# T(x\chi_B) \, d\mu \right|
$$

$$
\geqslant \left| \beta + \lambda \int_A x^\# T(x\chi_B) \, d\mu - (1+\lambda)^{p-1}\beta \right|
$$

$$
- (1+\lambda)^{p-1} \left| \int_\Omega x^\# Tx \, d\mu \right| - |\lambda|(1+\lambda)^{p-1} \left| \int_B x^\# T(x\chi_B) \, d\mu \right|
$$

$$
\geqslant \left| \left(1 - (1+\lambda)^{p-1}\right)\beta + \lambda \int_A x^\# T(x\chi_B) \, d\mu \right| - (1+|\lambda|)(1+\lambda)^{p-1} v(T).
$$

This, together with (4), gives us that

$$
(5) \qquad v(T)\left((1+|\lambda|)(1+\lambda)^{p-1} + \max\{1, (1+\lambda)^p\} \right) \geqslant \left| (1 - (1+\lambda)^{p-1})\beta + \lambda \int_A x^\# T(x\chi_B) d\mu \right|.
$$

Therefore, putting $a = \beta^{-1} \int_A x^\# T(x\chi_B) \, d\mu$ and

$$
f(\lambda) = |\lambda|^{-1}\left((1+|\lambda|)(1+\lambda)^{p-1} + \max\{1, (1+\lambda)^p\} \right) \qquad \left(\lambda \in [-1, \infty) \setminus \{0\} \right),
$$

and multiplying (5) by $|\lambda|^{-1}\beta^{-1}$, we obtain that

$$
\beta^{-1} v(T) f(\lambda) \geqslant \left| \frac{1 - (1+\lambda)^{p-1}}{\lambda} - a \right|
$$

for every $\lambda \in [-1, \infty) \setminus \{0\}$. Thus,

$$
\beta^{-1} v(T)\left(1 + f(\lambda)\right) = \beta^{-1} v(T)\left(f(-1) + f(\lambda)\right)
$$

$$
\geqslant \left| -1 - a \right| + \left| \frac{1 - (1+\lambda)^{p-1}}{\lambda} - a \right| \geqslant \left| \frac{(1+\lambda)^{p-1} - 1}{\lambda} - 1 \right|
$$

for every $\lambda \in [-1, \infty) \setminus \{0\}$ or, equivalently,

$$
v(T) \geqslant \beta \frac{\left| (1+\lambda)^{p-1} - 1 - \lambda \right|}{|\lambda| + \left(1 + |\lambda|\right)(1+\lambda)^{p-1} + \max\{1, (1+\lambda)^p\}}
$$

for every $\lambda \in [-1, \infty)$. Now we restrict ourselves to $\lambda \geqslant 0$ and setting $t = 1 + \lambda$, we obtain that

$$
v(T) \geqslant \beta \frac{|t^{p-1} - t|}{t - 1 + 2t^p} = \beta \frac{|t^{p-1} - t|}{1 + t^p} \frac{1 + t^p}{t - 1 + 2t^p}
$$

for every $t \in [1, \infty)$. Since it obviously holds that

$$
\frac{1 + t^p}{t - 1 + 2t^p} \geqslant \frac{1}{3}
$$

for each $t \in [1, \infty)$, one obtains that

$$
v(T) \geqslant \frac{\beta}{3} \sup_{t \geqslant 1} \frac{|t^{p-1} - t|}{1 + t^p} \geqslant \frac{|v|(T) - \varepsilon}{6} \sup_{t \geqslant 1} \frac{|t^{p-1} - t|}{1 + t^p} = \frac{|v|(T) - \varepsilon}{6} M_p,
$$

which is enough in view of the arbitrariness of ε. $\qquad\qquad \square$

Our next goal is to prove an inequality relating the absolute numerical radius and the norm of operators on real L_p-spaces.

Theorem 2. *Let $1 < p < \infty$ and let μ be a positive measure. Then, every bounded linear operator T on the real space $L_p(\mu)$ satisfies*

$$
|v|(T) \geqslant \frac{n\left(L_p^{\mathbb{C}}(\mu)\right)}{2} \|T\|,
$$

where $n\left(L_p^{\mathbb{C}}(\mu)\right)$ is the numerical index of the complex space $L_p(\mu)$.

Proof. We consider the complex linear operator $T_{\mathbb{C}} \in \mathcal{L}\big(L_p^{\mathbb{C}}(\mu)\big)$ given by

$$(6) \qquad T_{\mathbb{C}}(x) = T(\operatorname{Re} x) + i\,T(\operatorname{Im} x) \qquad \big(x \in L_p^{\mathbb{C}}(\mu)\big).$$

Evidently, $\|T\| \leqslant \|T_{\mathbb{C}}\|$. Now, consider any simple function $x = \sum_{j=1}^{m} a_j\, e^{i\theta_j}\, \chi_{A_j} \in L_p^{\mathbb{C}}(\mu)$ where $m \in \mathbb{N}$, $a_j \geqslant 0$, $\theta_j \in [0, 2\pi)$, the sets $A_1, \ldots, A_m \in \Sigma$ are pairwise disjoint, $\sum_{j=1}^{m} a_j^p \mu(A_j) = 1$, and observe that $x^{\#} \in L_q^{\mathbb{C}}(\mu)$ is given by the formula

$$x^{\#} = \sum_{j=1}^{m} a_j^{p-1}\, e^{-i\theta_j}\, \chi_{A_j}.$$

Then, writing

$$\alpha_{j,k} = \int_{A_j} T_{\mathbb{C}}(\chi_{A_k})\, d\mu = \int_{A_j} T(\chi_{A_k})\, d\mu,$$

we obtain that

$$\left| \int_{\Omega} x^{\#} T_{\mathbb{C}}(x)\, d\mu \right| = \left| \sum_{j=1}^{m} a_j^{p-1}\, e^{-i\theta_j} \sum_{k=1}^{m} a_k\, e^{i\theta_k}\, \alpha_{j,k} \right| \leqslant \sum_{j=1}^{m} a_j^{p-1} \left| \sum_{k=1}^{m} a_k\, e^{i\theta_k}\, \alpha_{j,k} \right|$$

$$(7) \qquad \leqslant \sum_{j=1}^{m} a_j^{p-1} \left(\left| \sum_{k=1}^{m} a_k \cos(\theta_k)\, \alpha_{j,k} \right| + \left| \sum_{k=1}^{m} a_k \sin(\theta_k)\, \alpha_{j,k} \right| \right)$$

$$\leqslant 2 \max_{(z_k) \in [-1,1]^m} \sum_{j=1}^{m} a_j^{p-1} \left| \sum_{k=1}^{m} a_k z_k \alpha_{j,k} \right| = 2 \max_{(z_k) \in \{-1,1\}^m} \sum_{j=1}^{m} a_j^{p-1} \left| \sum_{k=1}^{m} a_k z_k \alpha_{j,k} \right|,$$

where the last equality follows from the convexity of the function $f : [-1, 1]^m \longrightarrow \mathbb{R}$ defined by

$$f(z_1, \ldots, z_m) = \sum_{j=1}^{m} a_j^{p-1} \left| \sum_{k=1}^{m} a_k z_k \alpha_{j,k} \right|.$$

On the other hand, for any finite sequence $(z_k) \in \{-1, 1\}^m$, putting

$$y_{(z_k)} = \sum_{j=1}^{m} a_j z_j \chi_{A_j} \in L_p(\mu),$$

one has $\|y_{(z_k)}\| = 1$ and that

$$\int_{\Omega} \left| y_{(z_\ell)}^{\#} T(y_{(z_\ell)}) \right| d\mu = \int_{\Omega} \left| \sum_{j=1}^{m} a_j^{p-1} z_j \chi_{A_j} \sum_{k=1}^{m} a_k z_k T(\chi_{A_k}) \right| d\mu$$

$$= \sum_{j=1}^{m} \int_{A_j} \left| a_j^{p-1} z_j \sum_{k=1}^{m} a_k z_k T(\chi_{A_k}) \right| d\mu$$

$$= \sum_{j=1}^{m} a_j^{p-1} \int_{A_j} \left| \sum_{k=1}^{m} a_k z_k T \chi_{A_k} \right| d\mu$$

$$\geqslant \sum_{j=1}^{m} a_j^{p-1} \left| \int_{A_j} \sum_{k=1}^{m} a_k z_k T \chi_{A_k}\, d\mu \right| = \sum_{j=1}^{m} a_j^{p-1} \left| \sum_{k=1}^{m} a_k z_k \alpha_{j,k} \right|.$$

This, together with (7), implies that

$$2|v|(T) \geqslant 2 \max_{z_\ell \in \{-1,1\}} \int_{\Omega} \left| y_{(z_\ell)}^{\#} T(y_{(z_\ell)}) \right| d\mu$$

$$\geqslant 2 \max_{z_\ell \in \{-1,1\}} \sum_{j=1}^{m} a_j^{p-1} \left| \sum_{k=1}^{m} a_k z_k \alpha_{j,k} \right| \geqslant \left| \int_{\Omega} x^{\#} T_{\mathbb{C}}(x)\, d\mu \right|.$$

Since the set of all simple functions is dense in $L_p^{\mathbb{C}}(\mu)$, it follows from [1, Theorem 9.3] that the above inequality implies that

$$2|v|(T) \geqslant v(T_{\mathbb{C}}) \geqslant n\big(L_p^{\mathbb{C}}(\mu)\big)\|T_{\mathbb{C}}\| \geqslant n\big(L_p^{\mathbb{C}}(\mu)\big)\|T\|. \hspace{2cm} \square$$

It remains to notice that $n\big(L_p^{\mathbb{C}}(\mu)\big) \geqslant 1/\mathrm{e}$ (as happens for any complex Banach space, see [1, Theorem 4.1]), to get the following consequence from the above two theorems.

Corollary 3. *Let $1 < p < \infty$ and let μ be a positive measure. Then, in the real case, one has*

$$n\big(L_p(\mu)\big) \geqslant \frac{M_p}{12\,\mathrm{e}}$$

where $M_p = \max\limits_{t \geqslant 1} \dfrac{|t^{p-1} - t|}{1 + t^p}$.

Since, clearly, $M_p > 0$ for $p \neq 2$, we get the following consequence which answers in the positive a question raised by C. Finet and D. Li (see [5, 6]) also posed in [7, Problem 1].

Corollary 4. *Let $1 < p < \infty$, $p \neq 2$ and let μ be a positive measure. Then $n\big(L_p(\mu)\big) > 0$ in the real case. In other words, the numerical radius and the operator norm are equivalent on $\mathcal{L}\big(L_p(\mu)\big)$.*

It is a particular case of [6, Theorem 2.2] that $n\big(L_p(\mu)\big) = \inf\limits_{m} n(\ell_p^m)$ for every infinite-dimensional $L_p(\mu)$-space. For finite-dimensional spaces, $n(\ell_p^m) \leqslant n(\ell_p^2)$ for every $m \geqslant 2$ since ℓ_p^2 is an ℓ_p-summand on ℓ_p^m and we may use [9, Remark 2.a]. On the other hand, it is clear that $n(\ell_p^2) \leqslant M_p$ (since M_p is the numerical radius of a norm-one operator on the real ℓ_p^2, see [8, Lemma 2] for instance). It then follows that

$$n\big(L_p(\mu)\big) \leqslant M_p$$

for every $1 < p < \infty$ and every positive measure μ such that $\dim\big(L_p(\mu)\big) \geqslant 2$. We do not know whether the above inequality is actually an equality.

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