# POLYNOMIAL NUMERICAL INDICES OF BANACH SPACES WITH ABSOLUTE NORM

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ABSTRACT. We study when a Banach space with absolute norm may have polynomial numerical indices equal to one. In the real case, we show that a Banach space $X$ with absolute norm, which has the Radon-Nikodým property or is Asplund, satisfies $n^{(2)}(X) < 1$ unless it is one-dimensional. In the complex case, we show that the only Banach spaces $X$ with absolute norm and the Radon-Nikodým property which satisfy $n^{(2)}(X) = 1$ are the spaces $\ell_\infty^m$. Also, the only Asplund complex space $X$ with absolute norm which satisfies $n^{(2)}(X) = 1$ is $c_0(\Lambda)$.

## 1. INTRODUCTION

Let $X$ be a Banach space over a scalar field $\mathbb{K}$ ($=\mathbb{R}$ or $=\mathbb{C}$). We write $B_X$ for the closed unit ball, $S_X$ for the unit sphere, $X^*$ for the dual space, and $\mathbb{T}$ for the set of modulus-one scalars. We define $\Pi(X)$ to be the subset of $X \times X^*$ given by

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} \ : \ x^*(x) = 1\}.$$

For $k \in \mathbb{N}$, a bounded $k$-homogeneous polynomial $P : X \longrightarrow X$ is $P(x) = L(x, \ldots, x)$ for all $x \in X$, where $L : X \times \cdots \times X \longrightarrow X$ is a continuous $k$-multilinear map. We denote by $\mathcal{P}\left({}^k X; X\right)$ the space of all bounded $k$-homogeneous polynomials from $X$ into $X$ endowed with the norm

$$\|P\| = \sup\{\|P(x)\| \ : \ x \in B_X\}.$$

Given $P \in \mathcal{P}\left({}^k X; X\right)$, the *numerical range* of $P$ is the subset of the scalar field given by

$$V(P) = \{x^*(Px) \ : \ (x, x^*) \in \Pi(X)\},$$

and the *numerical radius* of $P$ is

$$v(P) = \sup\{|x^*(Px)| \ : \ (x, x^*) \in \Pi(X)\}.$$

In 2006, Y. S. Choi, D. García, S. G. Kim and M. Maestre [1] introduced the *polynomial numerical index of order $k$* of a Banach space $X$ as the constant $n^{(k)}(X)$ defined by

$$\begin{aligned} n^{(k)}(X) &= \max\left\{c \geqslant 0 \ : \ c\,\|P\| \leqslant v(P) \ \ \forall\, P \in \mathcal{P}\left({}^k X; X\right)\right\} \\ &= \inf\left\{v(P) \ : \ P \in \mathcal{P}\left({}^k X; X\right),\ \|P\| = 1\right\} \end{aligned}$$

for every $k \in \mathbb{N}$. This concept is a generalization of the *numerical index* of a Banach space (recovered for $k = 1$), first suggested by G. Lumer in 1968 (see [3]). For more information and background, we refer the reader to the survey paper [5] and to [1, 2, 4, 6, 7, 8] and references therein.

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Let us recall some facts about the polynomial numerical index which are relevant to our discussion. The easiest examples are $n^{(k)}(\mathbb{R}) = 1$ and $n^{(k)}(\mathbb{C}) = 1$ for every $k \in \mathbb{N}$. In the complex case, $n^{(k)}(C(K)) = 1$ for every $k \in \mathbb{N}$ and $n^{(2)}(\ell_1) \leqslant \frac{1}{2}$. The real spaces $\ell_1^m$, $\ell_\infty^m$, $c_0$, $\ell_1$ and $\ell_\infty$ have polynomial numerical index of order 2 equal to $1/2$. The inequality $n^{(k+1)}(X) \leqslant n^{(k)}(X)$ holds for every real or complex Banach space $X$ and every $k \in \mathbb{N}$, giving that $n^{(k)}(H) = 0$ for every $k \in \mathbb{N}$ and every real Hilbert space $H$ of dimension greater than one. On the other hand, $n^{(k)}(X) \geqslant k^{\frac{k}{1-k}}$ for every complex Banach space $X$ and every $k \geqslant 2$. The inequality $n^{(k)}(X^{**}) \leqslant n^{(k)}(X)$ holds for every real or complex Banach space $X$ and every $k \in \mathbb{N}$, and may be strict.

Our main goal in this paper is to extend two known results in the finite-dimensional case to infinite-dimensional Banach spaces with absolute norms (see the exact definition of an absolute norm in section 2 but, roughly speaking, it is a complete norm on a linear subspace of $\mathbb{K}^\Lambda$ which depends only on the modulus of the coordinates). In the complex case, it has been proved in [6] that the unique finite-dimensional Banach spaces $X$ with absolute norm satisfying that $n^{(2)}(X) = 1$ are the spaces $\ell_\infty^m$ for some $m \in \mathbb{N}$. We give two extensions of this result. Let $X$ be a complex Banach space $X$ with absolute norm satisfying $n^{(2)}(X) = 1$. If $X$ has the Radon-Nikodým property (RNP in short), then $X$ is isometric to $\ell_\infty^m$ for some $m$. If $X$ is an Asplund space, then $X = c_0(\Lambda)$ for some nonempty set $\Lambda$. In the real case, it was proved in [8] that there is no finite-dimensional real space with polynomial numerical index of order two equal to one. We extend this result to infinite-dimensional Banach spaces with absolute norm which have the RNP or are Asplund.

The outline of the paper is as follows. We present in section 2 the definitions and background on absolute norms. Section 3 is devoted to give a description of Banach spaces with absolute norm which have numerical index one. Finally, we give in section 4 the results on polynomial numerical indices commented above.

We finish this introduction with some needed notation. Given a nonempty set $\Lambda$, we write $\ell_\infty(\Lambda)$ to denote the Banach space of all bounded functions from $\Lambda$ to the base field endowed with the supremum norm. The Banach space $c_0(\Lambda)$ is the completion (actually the closure in $\ell_\infty(\Lambda)$) of the subspace of $\ell_\infty(\Lambda)$ consisting of all finitely valued functions from $\Lambda$ to $\mathbb{K}$. Equivalently, a function $x \in \ell_\infty(\Lambda)$ belongs to $c_0(\Lambda)$ if and only if for every $\varepsilon > 0$, the set $\{\lambda \in \Lambda : |x(\lambda)| \geqslant \varepsilon\}$ is finite. For $1 \leqslant p < \infty$, we write $\ell_p(\Lambda)$ for the Banach space of all functions $x : \Lambda \longrightarrow \mathbb{K}$ such that $\sum_{\lambda\in\Lambda} |x(\lambda)|^p$ is summable, endowed with the norm $\|x\| = \left[\sum_{\lambda\in\Lambda} |x(\lambda)|^p\right]^{\frac{1}{p}}$. When $\Lambda$ is infinite and countable, we just write $\ell_\infty = \ell_\infty(\Lambda)$, $c_0 = c_0(\Lambda)$ and $\ell_p = \ell_p(\Lambda)$. If $\Lambda$ has $m$-elements, we write $\ell_\infty^m = \ell_\infty(\Lambda) = c_0(\Lambda)$ and $\ell_p^m = \ell_p(\Lambda)$.

## 2. Preliminaries on absolute norms

Let $\Lambda$ be a nonempty set and let $X$ be a $\mathbb{K}$-linear subspace of $\mathbb{K}^\Lambda$ (the space of all functions from $\Lambda$ to the base field $\mathbb{K}$). An *absolute norm* on $X$ is a complete norm $\|\cdot\|_X$ satisfying

(a) Given $x, y \in \mathbb{K}^\Lambda$ with $|x(\lambda)| = |y(\lambda)|$ for every $\lambda \in \Lambda$, if $x \in X$, then $y \in X$ with $\|y\|_X = \|x\|_X$.
(b) For every $\lambda \in \Lambda$, the function $e_\lambda : \Lambda \longrightarrow \mathbb{K}$ given by $e_\lambda(\xi) = \delta_{\lambda\xi}$ for $\xi \in \Lambda$, belongs to $X$ with $\|e_\lambda\|_X = 1$.

We will write $\|\cdot\| = \|\cdot\|_X$ when the space $X$ is clear from the context.

**Remark 2.1.** The following two results can be deduced from the definition of absolute norm.

(c) $\ell_1(\Lambda) \subseteq X \subseteq \ell_\infty(\Lambda)$ with contractive inclusions. Equivalently,

$$\sup\{|x(\lambda)| \,:\, \lambda \in \Lambda\} \leqslant \|x\|_X \leqslant \sum_{\lambda\in\Lambda} |x(\lambda)| \qquad (x \in X).$$

(d) Given $x, y \in \mathbb{K}^\Lambda$ with $|y(\lambda)| \leqslant |x(\lambda)|$ for every $\lambda \in \Lambda$, if $x \in X$, then $y \in X$ with $\|y\|_X \leqslant \|x\|_X$.

By the sake of completeness, we include a sketch of the proof of the above remark which has been given to us by V. Kadets.

*Proof.* (c). It is straightforward from (b) that $\ell_1(\Lambda) \subseteq X$ with contractive inclusion. The other inclusion follows from the fact that given $x \in X$ and $A \subseteq \Lambda$, the function $y = x\chi_A$ belongs to $X$ and $\|y\|_X \leqslant \|x\|_X$, which can be easily proved by a convexity argument.

(d). We first observe that if $h \in \ell_\infty(\Lambda)$ takes only a finite number of values, then $h|x| \in X$ and $\|\, h|x| \,\|_X \leqslant \|h\|_\infty \|x\|_X$ (this can be deduced from (a) and (b) by a convexity argument and induction on the number of values of $h$). Now, we write $y = h|x|$ with $h \in [-1,1]^\Lambda$, take any sequence $(h_n)$ in $[-1,1]^\Lambda$ of functions taking a finite number of values such that $(h_n) \longrightarrow h$ uniformly (this exists thanks to Lebesgue approximation theorem) and consider the sequence $y_n = h_n|x|$. On the one hand, $y_n \in X$ with $\|y_n\|_X \leqslant \|x\|_X$ since $h_n$ takes a finite number of values. On the other hand,

$$\|y_n - y_m\|_X \leqslant \|h_n - h_m\|_\infty \|x\|_X \qquad (n, m \in \mathbb{N})$$

and so $(y_n)$ is a Cauchy sequence in $X$. As $X$ is complete, $(y_n)$ converges to some $z \in X$ with $\|z\|_X \leqslant \|x\|_X$. Finally, since convergence in $X$ forces uniform convergence, we have $y = z$. □

We write supp$(x)$ for the support of an element $x \in X$, i.e. $\operatorname{supp}(x) = \{\lambda \in \Lambda \,:\, x(\lambda) \neq 0\}$.

Examples of Banach spaces with absolute norm are $c_0(\Lambda)$, $\ell_p(\Lambda)$ for $1 \leqslant p \leqslant \infty$ and every Banach space with a one-unconditional basis, finite or infinite, viewed as subspace of $\mathbb{K}^m$ or $\mathbb{K}^\mathbb{N}$ via the basis.

Observe that a (real) Banach space $X \subset \mathbb{R}^\Lambda$ with absolute norm is a Banach lattice in the pointwise order. Actually, $X$ can be viewed as a Köthe space on the measure space $(\Lambda, \mathcal{P}(\Lambda), \nu)$ where $\nu$ is the counting measure on $\Lambda$, which is non-necessarily $\sigma$-finite, see [9] for background on Köthe spaces (over $\sigma$-finite measures). We say that $X$ is *order continuous* if $0 \leqslant x_\alpha \downarrow 0$ and $x_\alpha \in X$ imply that $\lim \|x_\alpha\| = 0$. This is known to be equivalent to the fact that $X$ does not contain an isomorphic copy of $\ell_\infty$ (since $X$ is order complete, see [9]). If $X$ is order continuous, the set of those functions with finite support is dense in $X$. Actually, for every $x \in X$, one has $x = \sum_{\lambda\in\Lambda} x(\lambda)\, e_\lambda$ in norm. If $\Lambda$ is countable, this exactly means that the set $\{e_\lambda \,:\, \lambda \in \Lambda\}$ is a one-unconditional basis. In the complex case, given a linear subspace $X \subset \mathbb{C}^\Lambda$ with absolute norm, we may consider the real part $\widetilde{X}$ of $X$ (just taking the real part of every function in $X$) which is a linear subspace of $\mathbb{R}^\Lambda$ with absolute norm (the restriction of the one in $X$) and apply all the definitions above to $\widetilde{X}$. Therefore, if $X$ is order continuous (i.e. it does not contain $\ell_\infty$), then the set of those functions with finite support is dense in $X$ and, actually, for every $x \in X$, one has $x = \sum_{\lambda\in\Lambda} x(\lambda)\, e_\lambda$ in norm, as in the real case.

The *Köthe dual* $X'$ of a Banach space $X \subset \mathbb{K}^\Lambda$ with absolute norm is the linear subspace of $\mathbb{K}^\Lambda$ defined by

$$X' = \left\{ y \in \mathbb{K}^\Lambda \;:\; \|y\|_{X'} := \sup_{x\in B_X} \sum_{\lambda\in\Lambda} |y(\lambda)||x(\lambda)| < \infty \right\}.$$

It is clear that the norm $\|\cdot\|_{X'}$ on $X'$ is absolute. Every element $y \in X'$ defines naturally a continuous linear functional on $X$ by the formula

$$x \longmapsto \sum_{\lambda\in\Lambda} y(\lambda)\, x(\lambda) \qquad \big(x \in X\big),$$

so we have $X' \subseteq X^*$ and this inclusion is isometric. For $\lambda \in \Lambda$, we will write $e'_\lambda$ for the functional provided by the function $e_\lambda \in X'$ when viewed as an element of $X^*$. If $X$ is order continuous, it is easy to prove using that functions with finite support are dense in $X$, that the inclusion $X' \subseteq X^*$ is surjective and so $X^*$ completely identified with the Banach space with absolute norm $X'$. Let us comment that this fact is well known for Köthe spaces defined on a $\sigma$-finite measure space and that the Radon-Nikodým theorem is used in the proof. In the case we are studying here, the measure spaces are not necessarily $\sigma$-finite, but since they are discrete, the Radon-Nikodým theorem is not needed and the proof of the fact that $X' = X^*$ is straightforward.

We give now elementary results on Banach spaces with absolute norm. We give their proof for the sake of completeness.

**Proposition 2.2.** *Let $\Lambda$ be a nonempty set and let $X$ be a linear subspace of $\mathbb{K}^\Lambda$ with an absolute norm.*

(a) *If there is an element $x_0 \in B_X$ such that $|x_0(\lambda)| = 1$ for every $\lambda \in \Lambda$, then $X = \ell_\infty(\Lambda)$ with equality of norms.*
(b) *If $\operatorname{ext}(B_{X^*}) \subseteq \{\omega\, e'_\lambda \,:\, \omega \in \mathbb{T},\, \lambda \in \Lambda\} \subseteq X'$, then $X = c_0(\Lambda)$ with equality of norms. In particular, if $X^* = X' = \ell_1(\Lambda)$, then $X = c_0(\Lambda)$.*

*Proof.* (a). Take $x \in \ell_\infty(\Lambda)$ and observe that $|x(\lambda)| \leqslant \big|\|x\|_\infty\, x_0(\lambda)\big|$ for every $\lambda \in \Lambda$ and that $\|x\|_\infty\, x_0 \in X$. Therefore, $x \in X$ and $\|x\|_X \leqslant \|x\|_\infty$. Since one always has that $X \subseteq \ell_\infty(\Lambda)$ and that $\|x\|_\infty \leqslant \|x\|_X$, we get that $X = \ell_\infty(\Lambda)$ with equality of norms.

(b). We first observe that, thanks to the Krein-Milman and Hahn-Banach theorems, for every $x \in X$, $\|x\|_X = \max\{|x^*(x)| \,:\, x^* \in \operatorname{ext}(B_{X^*})\}$. Since $e'_\lambda(x) = x(\lambda)$ for every $\lambda \in \Lambda$, it then follows that

$$\|x\|_X = \max\{|x(\lambda)| \,:\, \lambda \in \Lambda\} \tag{1}$$

for every $x \in X$. Therefore, $X \subseteq \ell_\infty(\Lambda)$ with equality of norms. Since functions with finite support are contained in $X$, which is complete, we obtain that $c_0(\Lambda) \subseteq X$. Let us prove the reversed inclusion. Suppose, for the sake of contradiction, that for an element $x \in X$ there is $\varepsilon > 0$ and an infinite subset $\Gamma$ of $\Lambda$ such that $|x(\lambda)| \geqslant \varepsilon$ for every $\lambda \in \Gamma$. Since $\Gamma$ is infinite, we may take a sequence $(\lambda_n) \subset \Gamma$ of different elements of $\Gamma$. Then, the function given by

$$y(\lambda_n) = \varepsilon\left(1 - \frac{1}{n}\right) \text{ for } n \in \mathbb{N} \quad \text{and} \quad y(\lambda) = 0 \text{ for } \lambda \notin \{\lambda_n \,:\, n \in \mathbb{N}\}$$

belongs to $X$ but the set $\{|y(\lambda)| \,:\, \lambda \in \Lambda\}$ has no maximum, a contradiction with (1). Therefore, for every $x \in X$ and every $\varepsilon > 0$, the set $\{\lambda \in \Lambda \,:\, |x(\lambda)| \geqslant \varepsilon\}$ is finite. This shows that $X \subseteq c_0(\Lambda)$, as desired. □

## 3. Banach spaces with absolute norm and numerical index one

The following characterization of real finite-dimensional Banach spaces with absolute norm which has numerical index one in terms of extreme points was given in [11] and extended to the complex case in [6]. We write $\operatorname{ext}(A)$ to denote the set of extreme points of a convex set $A$ and $\#[B]$ is the cardinal of a set $B$.

**Proposition 3.1.** [6, 11] *Let $X$ be $\mathbb{K}^m$ endowed with an absolute norm. Then, $n(X) = 1$ if and only if for every $x \in \operatorname{ext}(B_X)$ and every $x' \in \operatorname{ext}(B_{X^*})$,*

$$|x| \in \{0,1\}^m, \quad |x'| \in \{0,1\}^m \quad \textit{and} \quad \#\big[\operatorname{supp}(x) \cap \operatorname{supp}(x')\big] = 1.$$

Our goal in this section is to extend this result to some infinite-dimensional spaces. We start by proving some technical lemmas which will be useful in the sequel.

**Lemma 3.2.** *Let $\Lambda$ be a nonempty set and let $X$ be a linear subspace of $\mathbb{K}^\Lambda$ with absolute norm. If $x_0 \in S_X$ satisfies $|x^*(x_0)| = 1$ for all $x^* \in \operatorname{ext}(B_{X^*})$, then $|x_0| \in \{0,1\}^\Lambda$.*

*Proof.* We fix $\lambda \in \Lambda$, consider $x^* \in \operatorname{ext}(B_{X^*})$ such that $x^*(e_\lambda) = 1$ (such an extreme point exists thanks to the Kreim-Milman theorem) and write $y = x_0 - x_0(\lambda)e_\lambda$ to get $x_0 = x_0(\lambda)e_\lambda + y$. Now, for every $\omega \in \mathbb{T}$, consider the operator $\Phi_\omega : X \longrightarrow X$ given by

$$[\Phi_\omega(x)](\lambda) = \omega x(\lambda), \qquad [\Phi_\omega(x)](\xi) = x(\xi) \ \text{ if } \xi \neq \lambda.$$

Then $\Phi_\omega$ is an onto isometry on $X$ and so $\Phi_\omega^*(x^*) \in \operatorname{ext}(B_{X^*})$. Therefore,

$$1 = \big|[\Phi_\omega^*(x^*)](x_0)\big| = \big|x^*(\Phi_\omega(x_0))\big| = \big|x^*\big(\omega x_0(\lambda)e_\lambda + y\big)\big| = \big|\omega x_0(\lambda) + x^*(y)\big|.$$

Moving $\omega \in \mathbb{T}$, we get that $|x_0(\lambda)| \in \{0,1\}$ (indeed, it follows that $|x_0(\lambda)| + |x^*(y)| = 1$ and $\big||x_0(\lambda)| - |x^*(y)|\big| = 1$ and this clearly gives $|x_0(\lambda)|, |x^*(y)| \in \{0,1\}$). □

We need some notation for the second lemma. We write $\operatorname{conv}(A)$ to denote the convex hull of a subset $A$ of a Banach $X$ and $\overline{\operatorname{conv}}(A)$ denotes the closure of $\operatorname{conv}(A)$ in the norm topology of $X$. If $X$ is a dual space, $\overline{\operatorname{conv}}^{w^*}(A)$ denotes the closure of $\operatorname{conv}(A)$ in the weak$^*$-topology.

**Lemma 3.3.** *Let $\Lambda$ be a nonempty set and let $X$ be a linear subspace of $\mathbb{K}^\Lambda$ with absolute norm.*

(a) *Suppose that $X$ is order continuous and there is $A \subset S_X$ such that $B_X = \overline{\operatorname{conv}}(A)$ and $|x'(x)| = 1$ for every $x \in A$ and every $x' \in \operatorname{ext}(B_{X^*})$. Then*

$$|x|, |x'| \in \{0,1\}^\Lambda \ \text{ and } \ \#\big[\operatorname{supp}(x) \cap \operatorname{supp}(x')\big] = 1 \qquad \big(x \in A,\ x' \in \operatorname{ext}(B_{X^*})\big).$$

(b) *Suppose that both $X$ and $X^* = X'$ are order continuous and there is $A \subset S_{X^*}$ such that $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(A)$ and $|x''(x')| = 1$ for every $x' \in A$ and every $x'' \in \operatorname{ext}(B_{X^{**}})$. Then*

$$|x'|, |x''| \in \{0,1\}^\Lambda \ \text{ and } \ \#\big[\operatorname{supp}(x') \cap \operatorname{supp}(x'')\big] = 1 \qquad \big(x' \in A,\ x'' \in \operatorname{ext}(B_{X^{**}})\big).$$

*Proof.* (a). Fix $x \in A$ and $x' \in \operatorname{ext}(B_{X^*})$. It follows directly from Lemma 3.2 that $|x| \in \{0,1\}^\Lambda$. Since $B_X = \overline{\operatorname{conv}}(A)$, we have $B_{X^{**}} = \overline{\operatorname{conv}}^{w^*}(A)$ and so $\operatorname{ext}(B_{X^{**}}) \subseteq \overline{A}^{w^*}$ by the reversed Krein-Milman theorem. It then follows that $|x^{**}(x')| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and Lemma 3.2 gives $|x'| \in \{0,1\}^\Lambda$. Now, suppose that there are $\lambda, \xi \in \operatorname{supp}(x) \cap \operatorname{supp}(x')$ with $\lambda \neq \xi$. Then

$$|x(\lambda)| = |x(\xi)| = |x'(\lambda)| = |x'(\xi)| = 1$$

and, if we consider $y' \in X'$ given by $y'(\mu) = |x'(\mu)|\operatorname{sign}\big(\overline{x(\mu)}\big)$ for every $\mu \in \Lambda$, then $y' \in \operatorname{ext}(B_{X^*})$ (since the norm of $X'$ is absolute) and

$$|y'(x)| = \sum_{\mu \in \Lambda} |x'(\mu)||x(\mu)| \geqslant |x(\lambda)| + |x(\xi)| = 2,$$

a contradiction.

(b). Fix $x' \in A$ and $x'' \in \operatorname{ext}(B_{X^{**}})$. First, Lemma 3.2 gives directly that $|x'| \in \{0,1\}^\Lambda$. To see that $|x''| \in \{0,1\}^\Lambda$, we fix $\lambda \in \Lambda$ and take $x'_0 \in A$ such that $x'_0(\lambda) =: \theta \in \mathbb{T}$ (we may take such $x'_0$ since, otherwise, $x'(\lambda) = x'(e_\lambda) = 0$ for every $x' \in A$ and this contradicts the fact that $e'_\lambda \in B_{X^*} = \overline{\operatorname{conv}}^{w^*}(A)$). Now, we write $x'_0 = \theta e'_\lambda + y'$, for every $\omega \in \mathbb{T}$ consider the surjective isometry $\Phi_\omega \in L(X^*)$ defined by

$$[\Phi_\omega(x')](\lambda) = \omega x'(\lambda), \qquad [\Phi_\omega(x')](\xi) = x'(\xi) \ \text{ if } \xi \neq \lambda,$$

and use that $\Phi_\omega^*(x'') \in \operatorname{ext}(B_{X^{**}})$ to get

$$1 = \big|[\Phi_\omega^*(x'')](x'_0)\big| = \big|x''(\Phi_\omega(x'_0))\big| = \big|x''\big(\omega\theta e'_\lambda + y\big)\big| = \big|\omega\theta x''(\lambda) + x''(y')\big|.$$

Since $\theta \in \mathbb{T}$, moving $\omega \in \mathbb{T}$, we get that $|x''(\lambda)| \in \{0,1\}$.

Finally, to get that the intersection of the support has exactly one element, we argue as in item (a). □

Let us comment that Proposition 3.1 follows from the above lemma by using McGregor's characterization of finite-dimensional spaces with numerical index one (see [5, §3], for instance): a finite-dimensional space $X$ satisfies $n(X) = 1$ if and only if $|x^*(x)| = 1$ for every $x \in \operatorname{ext}(B_X)$ and every $x^* \in \operatorname{ext}(B_{X^*})$.

In the infinite-dimensional case we do not know of any characterization of Banach spaces with numerical index 1 which does not involve operators. Nevertheless, it is possible to get necessary conditions for a Banach space to have numerical index 1 similar to McGregor's characterization above, by using denting points and $w^*$-denting points (see [10] or [5, §3]). For Asplund spaces and for spaces having the RNP, these necessary conditions are also sufficient, and this allows us to use Lemma 3.3 to get two results similar to Proposition 3.1, one for spaces with the RNP and one for Asplund spaces, which we will use in the next section. We need some notation. Let $X$ be a Banach space. A *slice* of $B_X$ is the non-empty intersection of $B_X$ with an open half-space. A point $x \in S_X$ is said to be a *denting point* if its belongs to slices of $B_X$ of arbitrarily small diameter. If $X$ is a dual space and one can take slices with arbitrarily small diameter coming from $w^*$-open half-spaces, we say that $x$ is a *$w^*$-denting point*.

We start with the result for Banach spaces with the RNP.

**Theorem 3.4.** *Let $\Lambda$ be a nonempty set and let $X$ be a linear subspace of $\mathbb{K}^\Lambda$ with absolute norm. If $X$ has the Radon-Nikodým property (and so $X$ is order continuous), then the following are equivalent:*

- (*i*) $n(X) = 1$,
- (*ii*) $|x|, |x'| \in \{0,1\}^\Lambda$ *and* $\#\big[\operatorname{supp}(x) \cap \operatorname{supp}(x')\big] = 1$ *for every denting point* $x \in B_X$ *and every* $x' \in \operatorname{ext}(B_{X^*})$.

*Proof.* $X$ is order continuous since it has the RNP and so it does not contain $\ell_\infty$. Write $A$ to denote the set of denting points of $B_X$. Since $X$ has the RNP, $B_X = \overline{\operatorname{conv}}(A)$.

(*i*) ⇒ (*ii*). If $n(X) = 1$, it follows from [10, Lemma 1] that $|x'(x)| = 1$ for every $x \in A$ and every $x' \in \operatorname{ext}(B_{X^*})$. Then Lemma 3.3.a gives (*ii*).

(*ii*) ⇒ (*i*). It clearly follows from (*ii*) that $|x'(x)| = 1$ for every $x \in A$ and every $x' \in \operatorname{ext}(B_{X^*})$ and this implies $n(X) = 1$ for spaces with the RNP (since $\overline{\operatorname{conv}}(A) = B_X$), see [5, Proposition 6]. □

Here is the result for Asplund spaces.

**Theorem 3.5.** *Let $\Lambda$ be a nonempty set and let $X$ be a linear subspace of $\mathbb{K}^\Lambda$ with absolute norm. If $X$ is an Asplund space (and so $X$ and $X^* = X'$ are order continuous), then the following are equivalent:*

- (*i*) $n(X) = 1$,
- (*ii*) $|x'|, |x''| \in \{0,1\}^\Lambda$ *and* $\#\big[\operatorname{supp}(x') \cap \operatorname{supp}(x'')\big] = 1$ *for every $w^*$-denting point $x'$ of $B_{X^*}$ and every* $x'' \in \operatorname{ext}(B_{X^{**}})$.

*Proof.* $X$ and $X^*$ are order continuous since they do not contain $\ell_\infty$ (since $X$ is Asplund and so $X^*$ has the RNP). The proof of the equivalence is analogous to the one of Theorem 3.4, we give the details for the sake of completeness. We consider $A$ to be the set of $w^*$-denting points of $B_{X^*}$, and observe that $B_{X^*} = \overline{\text{conv}}^{w^*}(A)$ since $X$ is Asplund.

$(i) \Rightarrow (ii)$. If $n(X) = 1$, it follows from [10, Lemma 1] that $|x''(x')| = 1$ for every $x' \in A$ and every $x'' \in \text{ext}(B_{X^{**}})$. Then Lemma 3.3.b gives $(ii)$.

$(ii) \Rightarrow (i)$. It clearly follows from $(ii)$ that $|x''(x')| = 1$ for every $x' \in A$ and every $x'' \in \text{ext}(B_{X^{**}})$, and this implies $n(X) = 1$ for Asplund spaces (since $\overline{\text{conv}}^{w^*}(A) = B_{X^*}$), see [5, Proposition 6]. $\square$

Let us finish the section with the following remark.

**Remark 3.6.** Given a nonempty set $\Lambda$ and a linear subspace $X \subset \mathbb{K}^\Lambda$ with absolute norm, no matter $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, it is possible to define the real and the complex versions of $X$ as

$$X(\mathbb{R}) = \{x : \Lambda \longrightarrow \mathbb{R} \ : \ |x| \in X\}, \qquad X(\mathbb{C}) = \{x : \Lambda \longrightarrow \mathbb{C} \ : \ |x| \in X\}.$$

Then, $X(\mathbb{R})$ is a (real) linear subspace of $\mathbb{R}^\Lambda$ with absolute norm and $X(\mathbb{C})$ is a (complex) linear subspace of $\mathbb{C}^\Lambda$ with absolute norm. They are related since $X(\mathbb{C}) = X(\mathbb{R}) + iX(\mathbb{R})$. It follows from Theorems 3.4 and 3.5 that, for Asplund spaces and for spaces having the RNP, $n(X(\mathbb{C})) = 1$ if and only if $n(X(\mathbb{R})) = 1$. We do not know whether the same is true without the Asplund or RNP assumptions.

## 4. Polynomial numerical index

The aim of this section, which is the main aim of the paper, is to show that there are not too many spaces with absolute norm which have polynomial numerical index of order 2 equal to one among Asplund spaces or spaces with the RNP. The first result is a general sufficient condition for a Banach space with absolute norm to have polynomial numerical index smaller than one, from which we will deduce the main results of the paper.

**Proposition 4.1.** *Let $\Lambda$ be a nonempty set and let $X$ be a linear subspace of $\mathbb{K}^\Lambda$ with absolute norm. Suppose that there is a set $B$ dense in $S_X$ and a set $C \subset S_{X'}$ such that $|x'| \in \{0,1\}^\Lambda$ for every $x' \in C$ and satisfying that for every $x \in B$ there is $x' \in C$ with $x'(x) = 1$. If there exist $x_0^* \in B_{X^*}$ and $s, t \in \Lambda$ with $s \neq t$ such that $|x_0^*(e_s)| = |x_0^*(e_t)| = 1$, then $n^{(2)}(X) \leqslant 9/16$.*

*Proof.* We first observe that

$$|x(s)| + |x(t)| \leqslant \|x\| \qquad (x \in X). \tag{2}$$

Indeed, we take $\omega_1, \omega_2 \in \mathbb{T}$ such that

$$\omega_1\, x_0^*(e_s)\, x(s) = |x(s)| \quad \text{and} \quad \omega_2\, x_0^*(e_t)\, x(t) = |x(t)|$$

and observe that

$$\begin{aligned}\|x\| &\geqslant \|x(s)e_s + x(t)e_t\| = \|\omega_1 x(s)e_s + \omega_2 x(t)e_t\| \\ &\geqslant \left|x_0^*(\omega_1 x(s)e_s + \omega_2 x(t)e_t)\right| = |x(s)| + |x(t)|.\end{aligned}$$

Now, we define $P \in \mathcal{P}\left({}^k X; X\right)$ by

$$P(x) = \left(\frac{1}{2}x(s)^2 + \frac{3}{2}x(s)x(t)\right) e_s - \left(\frac{1}{2}x(t)^2 + \frac{3}{2}x(s)x(t)\right) e_t \qquad (x \in X),$$

and use (2) to get that $\|P\| \geqslant \|P(\frac{1}{2}(e_s + e_t)\| = \|\frac{1}{2}(e_s + e_t)\| \geqslant 1$ (actually, $\|P\| = 1$). We claim that $v(P) \leqslant 9/16$. Indeed, since $B$ is dense in $S_X$, we may use [12, Theorem 2.5] to get that

$$v(P) = \sup\left\{\left|x'\big(P(x)\big)\right| \ : \ x \in B,\, x' \in C,\, x'(x) = 1\right\}.$$

Pick $x \in B$ and $x' \in C$ such that $x'(x) = 1$, write $J = \operatorname{supp}(x')$ and $I = \operatorname{supp}(x)$, and write $x(k) = r_k \,\mathrm{e}^{i\theta_k}$ where $r_k = |x(k)|$ and $\theta_k \in [0, 2\pi[$ for all $k \in \Lambda$. Then

$$1 = x'(x) = \sum_{k \in I \cap J} x'(k)x(k) \leqslant \sum_{k \in I \cap J} |x'(k)||x(k)| \leqslant |x'|(|x|) \leqslant |||x'|||\,|||x||| \leqslant 1$$

and therefore, $x'(k)x(k) = |x'(k)||x(k)|$ for every $k \in I \cap J$. Since $|x'| \in \{0,1\}^\Lambda$, we get that $x'(k) = \mathrm{e}^{-i\theta_k}$ for every $k \in I \cap J$. Let us first suppose that $\{s,t\} \subset I \cap J$. Then

$$\begin{aligned} \big|x'\big(P(x)\big)\big| &= \left| \left[ \frac{1}{2} r_s^2 \,\mathrm{e}^{i2\theta_s} + \frac{3}{2} r_s r_t \,\mathrm{e}^{i\theta_t + i\theta_s} \right] \mathrm{e}^{-i\theta_s} - \left[ \frac{1}{2} r_t^2 \,\mathrm{e}^{i2\theta_t} + \frac{3}{2} r_s r_t \,\mathrm{e}^{i\theta_s + i\theta_t} \right] \mathrm{e}^{-i\theta_t} \right| \\ &= \left| \frac{1}{2} r_s^2 \,\mathrm{e}^{i\theta_s} + \frac{3}{2} r_s r_t \,\mathrm{e}^{i\theta_t} - \frac{1}{2} r_t^2 e^{i\theta_t} - \frac{3}{2} r_s r_t e^{i\theta_s} \right| \leqslant \left| \frac{1}{2} r_s^2 - \frac{3}{2} r_s r_t \right| + \left| \frac{3}{2} r_s r_t - \frac{1}{2} r_t^2 \right|. \end{aligned}$$

Since $0 \leqslant r_s + r_t \leqslant 1$ by (2), we get that $|x'(P(x))| \leqslant 1/2$. If $s \in I \cap J$ and $t \notin I \cap J$, we have

$$\big|x'\big(P(x)\big)\big| \leqslant \frac{1}{2} r_s^2 + \frac{3}{2} r_s r_t \leqslant \frac{9}{16}$$

where the last inequality follows again from the fact that $0 \leqslant r_s + t_t \leqslant 1$. Analogously, $|x'(P(x))| \leqslant 9/16$ if $s \notin I \cap J$ and $t \in I \cap J$, and clearly $|x'(P(x))| = 0$ if $s \notin I \cap J$ and $t \notin I \cap J$. Summarizing, we have shown that $v(P) \leqslant 9/16$ and $\|P\| \geqslant 1$, which gives $n^{(2)}(X) \leqslant 9/16$ as desired. □

We may now state the two main results of the paper.

**Theorem 4.2.** *Let $\Lambda$ be a nonempty set and let $X$ be a (complex) linear subspace of $\mathbb{C}^\Lambda$ with absolute norm such that $n^{(2)}(X) = 1$.*

(a) *If $X$ has the Radon-Nikodým property, then $\Lambda$ is finite and $X = \ell_\infty^m$ for some $m \in \mathbb{N}$.*
(b) *If $X$ is an Asplund space, then $X = c_0(\Lambda)$.*

*Proof.* (a). $X$ is order continuous since it has the RNP. We take $B = B_X$ and $C = \operatorname{ext}(B_{X^*})$, and use Theorem 3.4 (we have $n(X) = 1$ since $n^{(2)}(X) = 1$) to get that $|x|, |x'| \in \{0,1\}^\Lambda$ and $\#\big[\operatorname{supp}(x') \cap \operatorname{supp}(x)\big] = 1$ for every denting point $x$ in $B_X$ and every $x' \in C$. Therefore, $B$ and $C$ satisfy the hypotheses of Proposition 4.1. Pick now any denting point $x_0 \in S_X$ and any $\lambda \in \Lambda$. We may find $x' \in C$ such that $x'(\lambda) = x'(e_\lambda) = 1$ and we deduce that $x'(\mu) = x'(e_\mu) = 0$ for every $\mu \neq \lambda$ by Proposition 4.1. Therefore, $\operatorname{supp}(x') \cap \operatorname{supp}(x_0) \subseteq \{\lambda\}$ and, since $\#\big[\operatorname{supp}(x') \cap \operatorname{supp}(x_0)\big] = 1$ and $|x_0| \in \{0,1\}^\Lambda$, we get $|x_0(\lambda)| = 1$. We deduce from Proposition 2.2.a that $X = \ell_\infty(\Lambda)$ with equality of norms. Since $\ell_\infty(\Lambda)$ has the RNP if and only if $\Lambda$ is finite, we get $X = \ell_\infty^m$ for some $m \in \mathbb{N}$.

(b). First, $X$ and $X^* = X'$ are order continuous since $X$ is Asplund. We take $B$ to be the set of points in $S_X$ where the norm of $X$ is Fréchet differentiable and $C$ to be the set of $w^*$-denting points of $B_{X^*}$. Since $X$ is Asplund, $B$ is dense in $S_X$ and for every $x \in B$, the unique $x' \in B_{X^*}$ such that $x'(x) = 1$ belongs to $C$ (indeed, $x$ defines slices of $B_{X^*}$ of arbitrary small diameter containing $x'$). We also have $|x'| \in \{0,1\}^\Lambda$ for every $x' \in C$ by Theorem 3.4 (we have $n(X) = 1$ since $n^{(2)}(X) = 1$). Moreover, from Theorem 3.4 we deduce that $|x'|, |x''| \in \{0,1\}^\Lambda$ and $\#\big[\operatorname{supp}(x') \cap \operatorname{supp}(x'')\big] = 1$ for every $x' \in C$ and every $x'' \in \operatorname{ext}(B_{X^{**}})$. Pick now any $x_0'' \in \operatorname{ext}(B_{X^{**}})$ and any $\lambda \in \Lambda$. We may find $x' \in C$ such that $x'(\lambda) = x'(e_\lambda) \neq 0$ and since $|x'(\lambda)| \in \{0,1\}$, we get $|x'(e_\lambda)| = 1$ and we deduce that $x'(\mu) = x'(e_\mu) = 0$ for every $\mu \neq \lambda$ by Proposition 4.1 and the fact that $|x'| \in \{0,1\}^\Lambda$. Therefore, $\operatorname{supp}(x_0'') \cap \operatorname{supp}(x') \subseteq \{\lambda\}$ and, since $\#\big[\operatorname{supp}(x_0'') \cap \operatorname{supp}(x')\big] = 1$ and $|x_0''| \in \{0,1\}^\Lambda$, we get $|x_0''(\lambda)| = 1$. We then deduce from Proposition 2.2.a that $X'' = \ell_\infty(\Lambda)$ with equality of norms and, therefore, $X' = \ell_1(\Lambda)$ ($\ell_1(\Lambda)$ is the unique isometric predual of $\ell_\infty(\Lambda)$). Finally, Proposition 2.2.b gives us that $X = c_0(\Lambda)$ with equality of norms. □

**Theorem 4.3.** *Let $\Lambda$ be a nonempty set and let $X$ be a (real) linear subspace of $\mathbb{R}^\Lambda$ with absolute norm such that $n^{(2)}(X) = 1$. If $X$ has the Radon-Nikodým property or $X$ is an Asplund space, then $X = \mathbb{R}$.*

*Proof.* We may follow literally the proof of the complex case to get $X = \ell_\infty^m$ or $X = c_0(\Lambda)$ but, in the real case, $n^{(2)}(\ell_\infty^m) = 1/2$ if $m \geqslant 2$ and $n^{(2)}(c_0(\Lambda)) = 1/2$ if $\#\Lambda \geqslant 2$ [7, Corollary 2.5], Therefore, $X = \mathbb{R}$ as desired. □

## References


[1] Y. S. Choi, D. García, S. G. Kim and M. Maestre The polynomial numerical index of a Banach space, *Proc. Edinb. Math. Soc.* **49** (2006), 39–52. 1

[2] Y. S. Choi, D. García, M. Maestre, and M. Martín, Polynomial numerical index for some complex vector-valued function spaces, *Quart. J. Math.* **59** (2008), 455–474. 1

[3] J. Duncan, C. McGregor, J. Pryce, and A. White, The numerical index of a normed space, *J. London Math. Soc.* **2** (1970), 481–488. 1

[4] D. García, B. Grecu, M. Maestre, M. Martín and J. Merí, Two-dimensional Banach spaces with polynomial numerical index zero, *Linear Algebra Appl.* **430** (2009), 2488–2500. 1

[5] V. Kadets. M. Martín, and R. Payá, Recent progress and open questions on the numerical index of Banach spaces, *Rev. R. Acad. Cien. Serie A. Mat.* **100** (2006), 155–182. 1, 3, 3, 3

[6] J. Kim and H. J. Lee, Strong peak points and strongly norm attaining points with applications to denseness and polynomial numerical indices, *J. Funct. Anal.* **257** (2009), 931–947. 1, 3, 3.1

[7] S. G. Kim, M. Martín, and J. Merí, On the polynomial numerical index of the real spaces $c_0$, $\ell_1$ and $\ell_\infty$, *J. Math. Anal. Appl.* **337** (2008), 98–106. 1, 4

[8] H. J. Lee, Banach spaces with polynomial numerical index 1, *Bull. London Math. Soc.* **40** (2008), 193–198. 1

[9] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. II*, Springer–Verlag, Berlin–Heidelberg–New York, 1979. 2

[10] G. López, M. Martín, and R. Payá, Real Banach spaces with numerical index 1, *Bull. London Math. Soc.* **31** (1999), 207–212. 3, 3, 3

[11] S. Reisner, Certain Banach spaces associated with graphs and CL-spaces with 1-unconditonal bases, *J. London Math. Soc* **43** (1991), 137–148. 3, 3.1

[12] A. Rodríguez-Palacios, Numerical ranges of uniformly continuous functions on the unit sphere of a Banach space, *J. Math. Anal. Appl.* **2**97 (2004), no. 2, 472–476. 4



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