LUSHNESS, NUMERICAL INDEX ONE AND DUALITY

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Abstract. We construct an example of a Banach space which is not lush, but whose dual space is lush. This example shows that lushness is not equivalent to numerical index one. A characterization of lushness for some quotient spaces of $L_1(\mu)$ spaces and new results on C-rich subspaces of (scalar- or vector-valued) $C(K)$ spaces are also presented.

1. Introduction

Given a real or complex Banach space X, we write B_X, S_X, X^* and $L(X)$ to denote, respectively, the closed unit ball, the unit sphere, the topological dual and the Banach algebra of bounded linear operators on X. $\mathbb{T}$ denotes the set of modulus one scalars. For a bounded subset A of X, $x^* \in S_{X^*}$ and $\varepsilon > 0$, we write $S(A, x^*, \varepsilon)$ to denote the open slice

$$S(A, x^*, \varepsilon) = \{x \in S_X \,:\, \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \varepsilon\}.$$

If B is a subset of X, we write $\operatorname{aconv}(B)$ for the absolutely convex hull of B. Finally, we denote by $\operatorname{ext}(C)$ the set of extreme points of the convex subset $C \subset X$.

Definition 1.1 ([4, Definition 2.1])**.** A Banach space X is *lush* if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice $S = S(B_X, x^*, \varepsilon)$ with $x^* \in S_{X^*}$ such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon.$$

Lushness was recently introduced in [4] as a geometrical property of a Banach space which ensures that the space has numerical index one. We recall that a Banach space X is said to have *numerical index 1* if the equality

$$\|T\| = \sup\{|x^*(Tx)| \,:\, x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}$$

holds for every operator $T \in L(X)$ or, equivalently (see [7]), for every operator $T \in L(X)$ there is $\omega \in \mathbb{T}$ such that

$$\|\mathrm{Id} + \omega\, T\| = 1 + \|T\|.$$

We refer the reader to the survey paper [9] for more information and background on numerical index.

Lushness was used in [4] to show that there is a Banach space having numerical index one whose dual space does not share this property, a long standing open question in

Date: November 19th, 2008. Revised: January 19th, 2009. Second revision: March 11th, 2009.

2000 *Mathematics Subject Classification.* 46B04, 46B20, 47A12.

Key words and phrases. Banach space; numerical index; lushness; duality; C-rich subspaces; quotient spaces of L_1 spaces.

First author partially supported by Junta de Andalucía and FEDER grant P06-FQM-01438. Second and third authors partially supported by Spanish MEC and FEDER project no. MTM2006-04837 and Junta de Andalucía and FEDER grants FQM-185 and P06-FQM-01438. Fourth author partially supported by N. I. Akhiezer foundation.

the theory of numerical indices. It has been also used recently to construct a Banach space whose Lie-algebra is trivial but the Lie-algebra of its dual is infinite-dimensional [15]. Also, lushness helps to estimate the polynomial numerical index of some spaces in [6, 13] and it allows to show that every separable Banach space containing c_0 can be equivalently renormed to have numerical index one [3]. Some examples of lush spaces are $L_1(\mu)$-spaces and their isometric preduals, including $C(K)$ spaces, function algebras and finite-codimensional subspaces of $C[0,1]$. For more information and background on lush spaces we refer the reader to the already cited [4] and to [3, 8].

The concept of numerical index one is difficult to manage, since its definition needs to deal with all operators on the space and we do not know of any geometrical characterization. There are in the literature several geometrical sufficient conditions for numerical index one, the weakest of all of them is lushness (see [3, §2] and [8, §1] for a detailed account). Among the advantages of the concept of lushness is that it is separably determined, stable by ultraproduct and stable by taking "reasonable" finite unconditional sums [3], and it is not known whether these results are true for Banach spaces with numerical index one. It is known that lushness is equivalent to numerical index one for Asplund spaces and for spaces with the Radon-Nikodým property (see [3, §2]) and, even more, for the so-called slicely countably determined Banach spaces [1], a class of spaces recently introduced which includes spaces not containing a copy of ℓ_1.

It has been even asked in [9, Problem 15] whether every Banach space with numerical index one is lush. The main result of this paper is to show that this is not the case. Actually, we provide an example of a Banach space X which is not lush, but whose dual space is lush. Since when the dual space has numerical index one so does the space, the same example shows that lushness is not equivalent to numerical index one. This is the content of section 4. To do so, we characterize in section 3 a family of quotient spaces of $L_1(\mu)$ which are lush. Finally, some new results on C-rich subspaces of (scalar or vector-valued) $C(K)$ spaces are also obtained. Namely, in section 5 we show that if E is a lush space, then $C(K,E)$ is lush in both real and complex cases and, in the real case, the same is true for every C-rich subspace of $C(K,E)$; in section 6 we show that a subspace X of a real $C(K)$ space with K perfect is C-rich if (and only if) every subspace of $C(K)$ containing X is lush.

We start our exposition with some characterizations of lushness collected in section 2.

2. On some reformulations of lushness

The first result we are going to present is a slight modification of [3, Theorem 4.1] and gives a reformulation of lushness only in terms of the space. The modification follows from the immediate idea that in the definition of lushness (Definition 1.1) nothing changes if we allow y to be in the unit ball instead of being in the unit sphere. We recall that a subset G of the unit ball of the dual of a Banach space X is said to be *norming* for X if $\|x\| = \sup\{|\phi(x)| : \phi \in G\}$ for every $x \in X$ and G is *rounded* if $\mathbb{T}G = G$.

Proposition 2.1. *Let X be a Banach space and $G \subset S_{X^*}$ be a norming rounded subset. Then, the following are equivalent:*

(*i*) *X is lush.*

$(ii)_{\mathbb{R}}$ *In the real case: for every* $x \in S_X$, $y \in B_X$ *and* $\varepsilon > 0$, *there exist* $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$ *and* $x_1, x_2 \in B_X$ *such that*

$$\|x + x_1 + x_2\| > 3 - \varepsilon$$

and

$$\|y - (\lambda_1 x_1 - \lambda_2 x_2)\| < \varepsilon$$

$(ii)_{\mathbb{C}}$ *In the complex case: For every* $x \in S_X$, $y \in B_X$, $n \in \mathbb{N}$ *and* $\varepsilon > 0$, *there exist* $\lambda_1, \ldots, \lambda_n \geqslant 0$, $\sum_{k=1}^{n} \lambda_k = 1$ *and* $x_1, \ldots, x_n \in B_X$ *such that*

$$\left\| x + \sum_{k=1}^{n} x_k \right\| > n + 1 - \varepsilon$$

and

$$\left\| y - \sum_{k=1}^{n} \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) x_k \right\| < \varepsilon + \frac{2\pi}{n}$$

(iii) *For every* $x \in S_X$, $y \in B_X$ *and for every* $\varepsilon > 0$, *there is* $x^* \in G$ *such that* $x \in S = S(B_X, x^*, \varepsilon)$ *and* $\operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon$.

Proof. In the definition of lushness, given x and y in S_X, we build a slice S such that $x \in S$ and $\operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon$. But, of course, $\operatorname{dist}(\alpha y, \operatorname{aconv}(S)) < \varepsilon$ for every $\alpha \in [0, 1]$, i.e. from the very beginning we could take an arbitrary $y \in B_X$ instead of $y \in S_X$. With this in mind, we can follow the proof of [3, Theorem 4.1] to show that we can also take $y \in B_X$ in the assertions $(ii)_{\mathbb{R}}$, $(ii)_{\mathbb{C}}$ and (iii). This is exactly our statements. □

In the separable case, it is possible to give another characterization of lushness only in terms of the dual space. Let us comment that it is an easy reformulation of [8, Theorem 4.3].

Proposition 2.2. *For a* separable *Banach space* X*, the following are equivalent:*

(i) X *is lush.*
(ii) *There is a norming subset* $\tilde{K} \subset \operatorname{ext}(B_{X^*})$ *such that for every* $x_1^* \in \tilde{K}$ *and for every* $x_2^* \in S_{X^*}$ *there is* $\theta \in \mathbb{T}$ *such that* $\|x_1^* + \theta x_2^*\| = 2$.

Proof. By [14, Theorem 2.1] the property of $\tilde{K}$ in condition (ii) is equivalent to

$$|x^{**}(x^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{X^{**}}),\ x^* \in \tilde{K}\big).$$

Now, that separable lush spaces fulfill this condition is exactly what was proved in [8, Theorem 4.3] and the reversed result follows from [3, Theorem 2.1]. □

Let us finish the section with another reformulation of lushness only valid in the real case which appeared in [9, pp. 164] without proof. We do not know of any complex analogue of this result.

Proposition 2.3. *Let* X *be a real Banach space. Then, the following are equivalent:*

(i) X *is lush,*
(ii) *for every* $x \in S_X$, $y \in B_X$ *and every* $\varepsilon > 0$, *there are* $z \in S_X$, $\gamma_1, \gamma_2 \in \mathbb{R}$ *with* $|\gamma_1 - \gamma_2| = 2$, *such that*

$$\|x + z\| \geqslant 2 - \varepsilon \quad \text{and} \quad \|y + \gamma_i z\| \leqslant 1 + \varepsilon \quad (i = 1, 2).$$

Proof. $(i) \Rightarrow (ii)$. Given $x \in S_X$, $y \in B_X$ and $\varepsilon > 0$, we use the definition of lushness (actually Proposition 2.1) to take $x^* \in S_{X^*}$, $\lambda \in [0,1]$, and $x_1, x_2 \in S(B_X, x^*, \varepsilon/3)$ satisfying

$$x^*(x) > 1 - \frac{\varepsilon}{3} \qquad \text{and} \qquad \big\|y - \big(\lambda x_1 - (1-\lambda)x_2\big)\big\| < \frac{\varepsilon}{3}\,.$$

Now, it is easy to check that $z = \dfrac{x_1 + x_2}{\|x_1 + x_2\|}$, $\gamma_1 = 2 - 2\lambda$, and $\gamma_2 = -2\lambda$ fulfill the desired conditions.

$(ii) \Rightarrow (i)$. Fixed $x \in S_X$, $y \in B_X$ and $0 < \varepsilon < 1$, we take $0 < \delta < \varepsilon/2$ satisfying $\frac{1-3\delta}{1+\delta} > 1 - \varepsilon$. By hypothesis, we may find $\gamma_1, \gamma_2 \in \mathbb{R}$ and $z \in S_X$ such that $|\gamma_1 - \gamma_2| = 2$,

$$\|x + z\| \geqslant 2 - \delta, \quad \|y + \gamma_1 z\| \leqslant 1 + \delta \quad \text{and} \quad \|y + \gamma_2 z\| \leqslant 1 + \delta.$$

Then, there is $x^* \in S_{X^*}$ satisfying $x^*(x+z) \geqslant 2 - \delta$, so

$$x^*(x) \geqslant 1 - \delta \qquad \text{and} \qquad x^*(z) \geqslant 1 - \delta, \tag{1}$$

hence $x \in S(B_X, x^*, \varepsilon)$. On the other hand, using the fact that $\|y + \gamma_1 z - \gamma_2 z - y\| = 2$, we obtain

$$1 - \delta \leqslant \|y + \gamma_1 z\| \leqslant 1 + \delta \quad \text{and} \quad 1 - \delta \leqslant \|y + \gamma_2 z\| \leqslant 1 + \delta.$$

We define

$$x_1 = \frac{y + \gamma_1 z}{\|y + \gamma_1 z\|} \qquad \text{and} \qquad x_2 = \frac{-y - \gamma_2 z}{\|y + \gamma_2 z\|}$$

and, using (1) and the fact that $\|y + \gamma_1 z\| x_1 + \|y + \gamma_2 z\| x_2 = 2z$, we deduce that

$$\|y + \gamma_1 z\| x^*(x_1) + \|y + \gamma_2 z\| x^*(x_2) = 2x^*(z) \geqslant 2 - 2\delta,$$

which implies

$$x^*(x_1) \geqslant \frac{1 - 3\delta}{1 + \delta} \qquad \text{and} \qquad x^*(x_2) \geqslant \frac{1 - 3\delta}{1 + \delta}$$

and, therefore, $x_1, x_2 \in S(B_X, x^*, \varepsilon)$.

Finally, if $\gamma_1 > 0$ and $\gamma_2 < 0$, we have that

$$\begin{aligned} \big\|y - \big(\tfrac{-\gamma_2}{2}\, x_1 - \tfrac{\gamma_1}{2}\, x_2\big)\big\| &= \big\|\tfrac{-\gamma_2}{2}\,(y + \gamma_1 z) + \tfrac{\gamma_1}{2}\,(y + \gamma_2 z) - \tfrac{-\gamma_2}{2}\, x_1 + \tfrac{\gamma_1}{2}\, x_2\big\| \\ &\leqslant \tfrac{-\gamma_2}{2}\,\big|1 - \|y + \gamma_1 z\|\big| + \tfrac{\gamma_1}{2}\,\big|1 - \|y + \gamma_2 z\|\big| < \delta < \varepsilon, \end{aligned}$$

and analogously if $\gamma_1 < 0$ and $\gamma_2 > 0$. Otherwise, there is $i \in \{1, 2\}$ such that $|\gamma_i| \leqslant \delta$ and then $\|y - (-1)^i x_i\| \leqslant \big|1 - \|y + \gamma_i z\|\big| + \delta \leqslant 2\delta < \varepsilon$. □

3. Lushness for some quotient spaces of $L_1(\mu)$

The aim of this section is to characterize some quotient spaces of $L_1(\mu)$ which are lush. The construction uses duality argument, so we start with some statements about $L_\infty(\mu)$.

We recall that a subspace X of a $C(K)$ space is said to be *C-rich* [4, Definition 2.3] if for every nonempty open subset U of K and every $\varepsilon > 0$, there is a positive function $h \in C(K)$ of norm 1 with support inside U such that the distance from h to X is less than ε. It follows from [2, Proposition 4.2] (and in the real case for $K = [0,1]$ it was proved in [10, Lemma 1.4.]), that the positivity of the function h in this definition can be omitted. A C-rich subspace of $C(K)$ is necessarily lush [4, Theorem 2.4].

Considering an $L_\infty(\Omega, \Sigma, \mu)$ space as a $C(K_\mu)$ space (where K_μ is the space of maximal ideals of the Banach algebra L_∞), the above definition applies to subspaces of

$L_\infty(\Omega,\Sigma,\mu)$. The following easy lemma gives a characterization of C-richness of a subspace of $L_\infty(\Omega,\Sigma,\mu)$ in terms of the L_∞ space itself.

Lemma 3.1. *Let (Ω,Σ,μ) be a measure space and let X be a subspace of $L_\infty(\Omega,\Sigma,\mu) \cong C(K_\mu)$. If for every subset $U \in \Sigma^+$ and every $\varepsilon > 0$ there is a function $h \in L_\infty(\Omega,\Sigma,\mu)$ of norm 1 with $\operatorname{supp}(h) \subset U$ and $\operatorname{dist}(h,X) < \varepsilon$, then X is C-rich.*

Proof. Observe that C-richness in $C(K_\mu)$ requires to find, for each open subset U of K_μ, a certain continuous function with support inside U, and the condition of the lemma only requires this for clopen subsets (this follows from the construction of K_μ). Since K_μ is extremally disconnected, each non-void open set contains a non-void clopen set, which proves the lemma. □

Recall that $\operatorname{ext}(B_{L_\infty(\Omega,\Sigma,\mu)})$ is the set of measurable functions on Ω which take almost everywhere modulus-one values. We call such functions "modulus-one functions".

We are now able to present the main result of this section.

Theorem 3.2. *Let (Ω,Σ,μ) be a finite measure space such that $L_1 := L_1(\Omega,\Sigma,\mu)$ is separable and consider $L_\infty := L_\infty(\Omega,\Sigma,\mu)$ as the dual space for L_1. Let $Y \subset L_1$ be a subspace whose annihilator $Y^\perp$ is C-rich in L_∞. Then, $X = L_1/Y$ is lush if and only if $G = Y^\perp \cap \operatorname{ext}(B_{L_\infty})$ is a norming subset of S_{X^*} for X.*

Proof. Let us first observe that X is separable since L_1 is. Now, if G is a norming subset of S_{X^*}, then $\tilde{K} = G$ evidently satisfies condition (ii) of Proposition 2.2 and X is lush. Conversely, suppose X is lush and let $\tilde{K} \subset X^* = Y^\perp$ be the norming subset from Proposition 2.2.(ii). Let us prove for every $f \in \tilde{K}$ the condition $|f| = 1$ a.e., which will ensure that $G \supset \tilde{K}$ and, consequently, that G is norming for X. Assume to the contrary that for some $f \in \tilde{K}$ there exists $\varepsilon > 0$ such that the set $A = \{t \in \Omega \,:\, |f(t)| < 1-\varepsilon\}$ has positive measure. Since $Y^\perp$ is C-rich in L_∞, there is $g \in S_{Y^\perp}$ such that $|g(t)| < \varepsilon$ for almost every $t \in \Omega \setminus A$. Then, the set

$$\{t \in \Omega \,:\, |f(t)| + |g(t)| > 2-\varepsilon\}$$

has measure zero and, therefore, $\|f + \theta g\| < 2 - \varepsilon$ for every $\theta \in \mathbb{T}$, which contradicts the definition of $\tilde{K}$. □

4. The main example

Let us present the announced construction of a non lush space whose dual is lush.

Consider $\Omega = [0,2]$ equipped with the standard Lebesgue measure. Introduce a partition $\Omega = \bigsqcup_{n=0}^{\infty} \Delta_n$ into subsets of positive measure with $\Delta_0 = [0,1]$. We consider all $L_\infty(\Delta_n)$ (in the natural way) as subspaces of $L_\infty[0,2]$. We denote by $\mathcal{F}$ the subspace of $L_\infty[1,2]$, consisting of the functions satisfying the condition

$$\int_{\Delta_n} f \, d\lambda = 0 \qquad (n \in \mathbb{N}).$$

For a fixed dense countable subset $\{f_m \,:\, m \in \mathbb{N}\} \subset S_{L_2[0,1]}$, let us define an operator $J : L_\infty[0,1] \longrightarrow L_\infty[1,2]$ as follows:

$$J(g) = \sum_{m\in\mathbb{N}} \left(\int_{[0,1]} g f_m \, d\lambda \right) \mathbf{1}_{\Delta_m} \qquad \big(g \in L_\infty[0,1]\big).$$

Observe that for every $g \in L_\infty[0,1]$ one has

$$\|J(g)\| = \sup_{m\in\mathbb{N}} \left| \int_{[0,1]} g\, f_m \; d\lambda \right| = \|g\|_{L_2[0,1]},$$

so J is a weakly compact operator mapping every modulus-one function from $L_\infty[0,1]$ into a norm-one element of $L_\infty[1,2]$. Finally, denote

$$\mathcal{Z} = \big\{ g + 2J(g) + f \ : \ g \in L_\infty[0,1], \ f \in \mathcal{F} \big\}.$$

Theorem 4.1. *$\mathcal{Z}$ is a weak*-closed C-rich subspace of $L_\infty[0,2]$, which does not contain any modulus-one function. Consequently, for $\mathcal{Y} = {}^\perp\mathcal{Z} \subset L_1[0,2]$, the quotient $\mathcal{X} = L_1[0,2]/\mathcal{Y}$ is a Banach space which is not lush, but whose dual $\mathcal{X}^* = \mathcal{Z}$ is lush.*

Proof. At first remark that $\mathcal{Z}$ can be written as the set of those $h \in L_\infty[0,2]$, for which the system of linear equations

$$2\int_{[0,1]} h f_m \, d\lambda = \frac{1}{\lambda(\Delta_m)} \int_{\Delta_m} h \, d\lambda \qquad (m \in \mathbb{N})$$

is valid. Since the left-hand and right-hand sides of all these equations are weak* continuous, the solution of the system is a weak*-closed linear subspace.

To demonstrate C-richness of $\mathcal{Z}$, let us fix a subset $A \subset [0,2]$ of positive measure and $\varepsilon > 0$. There are two (not mutually excluding) cases.
Case 1: $\lambda(A \cap [0,1]) > 0$. Then, since J is a weakly compact operator, its restriction on $L_\infty(A \cap [0,1])$ is also weakly compact, so for every $\varepsilon > 0$ there is a $g \in S_{L_\infty(A\cap[0,1])}$ with $\|Jg\| < \varepsilon/2$, so an element $g + 2J(g) \in \mathcal{Z}$ is the element whose distance from g is less than ε.
Case 2: $\lambda(A \cap [1,2]) > 0$. In this case there is $n \in \mathbb{N}$ for which $\lambda(A \cap \Delta_n) > 0$. One can evidently find $f \in S_{L_\infty(A\cap\Delta_n)}$ with $\int_{\Delta_n} f \, d\lambda = 0$. Then, we have $f \in \mathcal{F} \subset \mathcal{Z}$ and $\operatorname{supp}(f) \subset A$, which completes the proof of C-richness.

Finally, assume for the sake of contradiction that $\mathcal{Z}$ contains a modulus-one function $h = g + 2J(g) + f$ where $g \in L_\infty[0,1]$ and $f \in \mathcal{F}$. Since the supports of $J(g)$ and f lie in $[1,2]$, g is a modulus-one function on $[0,1]$. By definition of J, this means that

$$\sup_n \left| \frac{1}{\lambda(\Delta_n)} \int_{\Delta_n} J(g) \, d\lambda \right| = \|J(g)\| = 1,$$

so there is an $n \in \mathbb{N}$ for which

$$\frac{1}{\lambda(\Delta_n)} \left| \int_{\Delta_n} J(g) \, d\lambda \right| > \frac{1}{2}.$$

Consequently,

$$1 = \|h\| \geqslant \frac{1}{\lambda(\Delta_n)} \left| \int_{\Delta_n} 2J(g) \, d\lambda \right| > 1,$$

which is a contradiction. This shows, using Theorem 3.2, that $\mathcal{X}$ is not lush.

Finally, $\mathcal{X}^* = \mathcal{Z}$ is C-rich in $L_\infty[0,2]$ and so lush by [4, Theorem 2.4]. □

We now enunciate other properties of the space $\mathcal{X}$ constructed in the above theorem.

Remarks 4.2. Let $\mathcal{X}$ be the space constructed in Theorem 4.1.

(a) $\mathcal{X}$ *has numerical index one but it is not lush. This solves in the negative* [9, Problem 15]. Indeed, since $\mathcal{X}^*$ is lush, it follows that $\mathcal{X}^*$ has numerical index one [4, Proposition 2.1] and so does $\mathcal{X}$ (see [9, §2], for instance).

(b) It was asked in [9, Problem 13] and in [4, Remark 3.5], whether for every Banach space E with numerical index one, the subset of S_{E^*} given by

$$\mathcal{A}(E) = \{x^* \in S_{E^*} \,:\, |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \operatorname{ext}(B_{E^{**}})\}$$

is norming for E. Since this condition implies lushness of E [3, Theorem 2.1], *we have that* $\mathcal{A}(\mathcal{X})$ *is not norming and our space* $\mathcal{X}$ *answers in the negative the cited question.*

(c) Even more, *the set* $\mathcal{A}(\mathcal{X})$ *is empty.* Indeed, if $x^* \in \mathcal{A}(\mathcal{X})$, it is clear (using the Krein-Milman theorem for $B_{X^{**}}$) that for every $y^* \in S_{\mathcal{X}^*}$ there is $\theta \in \mathbb{T}$ such that $\|x^* + \theta y^*\| = 2$ (see [14, Theorem 2.1] for instance). Now, since $\mathcal{X}^* = \mathcal{Z}$ is a C-rich subspace of $L_\infty[0,2]$, it follows that x^* is a modulus-one function on $[0,2]$ (see the proof of Theorem 3.2). But $\mathcal{Z}$ does not contain any of such functions as it is proved in Theorem 4.1.

Let X be a Banach space. We have shown here that lushness of X^* does not imply lushness of X. Also, lushness of X does not imply lushness of X^* nor lushness of X^{**} (just consider X as the lush space presented in [3, Example 3.1] such that X^* and X^{**} do not have numerical index one). Therefore, the following result gives the unique true implication between the lushness of a space and lushness of the dual or of the bidual.

Proposition 4.3. *Let X be a Banach space. If X^{**} is lush, then X is lush.*

Proof. We fix $x, y \in S_X$ and $\varepsilon > 0$. Since X^{**} is lush and $S_{X^*} \subset S_{X^{***}}$ is norming for X^{**}, Proposition 2.1.(iii) allows us to find $z^* \in S_{X^*}$ such that

$$x \in S(B_{X^{**}}, z^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\big(y, \operatorname{aconv}\big(S(B_{X^{**}}, z^*, \varepsilon)\big)\big) < \varepsilon.$$

The first assertion above obviously implies that $x \in S(B_X, z^*, \varepsilon)$. The second assertion is equivalent to

$$y \in \operatorname{aconv}\big(S(B_{X^{**}}, z^*, \varepsilon)\big) + \varepsilon\, B_{X^{**}}$$

and so, since $\operatorname{aconv}\big(S(B_X, z^*, \varepsilon)\big)$ is weak*-dense in $\operatorname{aconv}\big(S(B_{X^{**}}, z^*, \varepsilon)\big)$ and $\varepsilon\, B_X$ is weak*-dense in $\varepsilon\, B_{X^{**}}$, we have

$$y \in \overline{\operatorname{aconv}\big(S(B_X, z^*, \varepsilon)\big) + \varepsilon\, B_X}^{w^*}.$$

Since $y \in X$, we can replace the weak*-closure above by weak closure, and so by norm closure by convexity, to get

$$\operatorname{dist}\big(y, \operatorname{aconv}\big(S(B_X, z^*, \varepsilon)\big)\big) < \varepsilon. \qquad \square$$

5. Lushness in vector-valued $C(K)$ spaces

Our first goal in this section is to show that lushness of the range space passes to every space of continuous functions. Let us comment that this result is known for Banach spaces with numerical index one [16, Theorem 5].

Proposition 5.1. *Let E be a lush Banach space and K be a Hausdorff compact. Then, the (real or complex) space $C(K,E)$ is also lush.*

Proof. We work only in the more difficult complex case, the real case being completely analogous. According to Proposition 2.1, it is enough to show that for every $f \in S_{C(K,E)}$, $g \in B_{C(K,E)}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist $\lambda_1, \ldots, \lambda_n \geqslant 0$, $\sum_{k=1}^n \lambda_k = 1$ and $f_1, \ldots, f_n \in B_{C(K,E)}$ such that:

$$\left\| f + \sum_{k=1}^n f_k \right\| > n + 1 - \varepsilon, \tag{2}$$

$$\left\| g - \sum_{k=1}^n \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) f_k \right\| \leqslant \varepsilon + \frac{2\pi}{n}. \tag{3}$$

Since $\|f\| = 1$, we can find $t_0 \in K$ such that $\|f(t_0)\| = 1$. So, we apply again Proposition 2.1 to $x = f(t_0) \in S_E$ and $y = g(t_0) \in B_E$. Then, we get $x_1, \ldots, x_n \in B_E$ and $\lambda_1, \ldots, \lambda_n \geqslant 0$, $\sum_{k=1}^n \lambda_k = 1$, such that

$$\left\| f(t_0) + \sum_{k=1}^n x_k \right\| > n + 1 - \varepsilon, \qquad \left\| g(t_0) - \sum_{k=1}^n \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) x_k \right\| \leqslant \frac{\varepsilon}{2} + \frac{2\pi}{n}.$$

Since g is continuous on K, we may find an open set $U \subset K$ such that $t_0 \in U$ and

$$\|g(t) - g(t_0)\| \leqslant \frac{\varepsilon}{2} \qquad (t \in U),$$

and a continuous function $\alpha : K \longrightarrow [0,1]$ with $\alpha(t_0) = 0$ and $\alpha|_{K \setminus U} \equiv 1$. If we define

$$f_k(t) = x_k + \alpha(t)\ \left(\exp\left(\tfrac{-2\pi i k}{n}\right) g(t) - x_k\right) \in B_{C(K,E)} \qquad \big(t \in K,\ k = 1, \ldots, n\big),$$

then the functions f_k satisfy conditions (2) and (3). Indeed, the fulfilment of (2) follows by just evaluating at t_0. For the second inequality, we consider an arbitrary $t \in K$ and we observe that

$$\left\| g(t) - \sum_{k=1}^n \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) f_k(t) \right\| = \left\| \big(1 - \alpha(t)\big) \left(g(t) - \sum_{k=1}^n \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) x_k \right) \right\|.$$

Now, if $t \in U$, we have $\|g(t) - g(t_0)\| \leqslant \dfrac{\varepsilon}{2}$ and so

$$\begin{aligned} \left\| g(t) - \sum_{k=1}^n \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) f_k(t) \right\| &\leqslant \|g(t) - g(t_0)\| + \left\| g(t_0) - \sum_{k=1}^n \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) x_k \right\| \\ &\leqslant \varepsilon + \frac{2\pi}{n}. \end{aligned}$$

If, otherwise, $t \notin U$, then $\alpha(t) = 1$ and so $g(t) - \sum_{k=1}^n \lambda_k \exp\left(\tfrac{2\pi i k}{n}\right) f_k(t) = 0$. ☐

The following definition follows the spirit of [2, Proposition 4.2] and it was stated in [5, Definition 2.3]. Recall that for $\alpha \in C(K)$ and $x \in E$, $\alpha \otimes x \in C(K,E)$ denotes the function $t \longmapsto \alpha(t)x$.

Definition 5.2. Let K be a compact space and let E be a Banach space. A subspace X of $C(K,E)$ is called C-rich if for every $\varepsilon > 0$, every $x \in E$ and every open subset U of K, there exists a nonnegative function $\alpha \in C(K)$ with $\|\alpha\| = 1$ and $\operatorname{supp}(\alpha) \subset U$ such that $\operatorname{dist}(\alpha \otimes x, X) < \varepsilon$.

The next result shows that C-rich subspaces are lush also in the vector-valued case. Since we will use Proposition 2.3, the proof is valid in the real case only.

Proposition 5.3. *Let E be a lush real Banach space and K be a Hausdorff compact space. Then, every C-rich subspace X of $C(K,E)$ is also lush.*

Proof. We fix $f \in S_X$, $g \in B_X$ and $0 < \varepsilon < 1$, and we find $t_0 \in K$ such that $\|f(t_0)\| = 1$. Since E is lush, we may use Proposition 2.3 to find $z \in S_X$, $\gamma_1, \gamma_2 \in \mathbb{R}$ with $|\gamma_1 - \gamma_2| = 2$ such that

$$\|f(t_0) + z\| \geqslant 2 - \varepsilon/5 \quad \text{and} \quad \|g(t_0) + \gamma_i z\| \leqslant 1 + \varepsilon/5.$$

We also take an open subset U of K containing t_0 such that

$$\|f(t_1) - f(t_2)\| < \varepsilon/5, \quad \|g(t_1) - g(t_2)\| < \varepsilon/5 \qquad (t_1, t_2 \in U).$$

As X is C-rich, we may find a nonnegative norm-one $\alpha \in C(K)$ and $h \in S_X$ such that

$$\operatorname{supp}(\alpha) \subset U \quad \text{and} \quad \|h - \alpha \otimes z\| < \varepsilon/5.$$

Let us show that h, γ_1 and γ_2 fulfil the conditions of Proposition 2.3, i.e.

$$\|f + h\| \geqslant 2 - \varepsilon \quad \text{and} \quad \|g + \gamma_i h\| \leqslant 1 + \varepsilon \ \ (i = 1, 2).$$

Indeed, for the first inequality we take $t_1 \in U$ such that $\alpha(t_1) = 1$ and observe that

$$\|f + h\| \geqslant \|f(t_1) + h(t_1)\| \geqslant \|f(t_0) + z\| - \|f(t_0) - f(t_1)\| - \|h(t_1) - z\| \geqslant 2 - \varepsilon.$$

To deal with the second inequality, we observe that

$$\begin{aligned}\|g(t) + \gamma_i h(t)\| &\leqslant \|g(t) + \gamma_i\, \alpha(t)\, z\| + |\gamma_i| \|\alpha(t) z - h\| \\ &\leqslant \|g(t) + \gamma_i\, \alpha(t)\, z\| + 3\varepsilon/5.\end{aligned}$$

If $t \notin U$, $\alpha(t) = 0$ and so $\|g(t) + \gamma_i h(t)\| \leqslant 1 + 3\varepsilon/5 \leqslant 1 + \varepsilon$. If $t \in U$, we have

$$\|g(t) + \gamma_i\, \alpha(t)\, z\| \leqslant \|g(t) - g(t_0)\| + \|g(t_0) + \gamma_i\, \alpha(t)\, z\| \leqslant 1 + 2\varepsilon/5,$$

since $g(t_0) + \gamma_i\, \alpha(t)\, z = (1 - \alpha(t))g(t_0) + \alpha(t)\big(g(t_0) + \gamma_i\, z\big) \in (1 + \varepsilon/5)B_X$. Then,

$$\|g(t) + \gamma_i h(t)\| \leqslant 1 + 2\varepsilon/5 + 3\varepsilon/4 = 1 + \varepsilon. \qquad \square$$

6. A characterization of C-rich subspaces of real $C(K)$ in terms of lushness

In [12] a general notion of richness, generated by the Daugavet property was introduced. We recall that a Banach space X has the *Daugavet property* whenever $\|\mathrm{Id} + T\| = 1 + \|T\|$ for every rank-1 operator $T \in L(X)$ [11]. If X is a Banach space with the Daugavet property, we say that a subspace Y of X is said to be *wealthy* if every subspace $Z \subset X$ containing Y has the Daugavet property. This concept was introduced in [12, §5] and it is equivalent to a concept of richness also introduced in the same paper [12, §5]. It is shown in [12] that all finite-codimensional subspaces of a space X with the Daugavet property are wealthy, and moreover, if X/Y does not contain copies of ℓ_1, or if X/Y has the RNP, then Y is wealthy. If K is a perfect compact space, wealth is equivalent to C-richness for subspaces of $C(K)$ [12].

Following these ideas, one can introduce an analogous concept for lushness.

Definition 6.1. Let X be a lush Banach space. A subspace Y of X is said to be *lush-wealthy* if every subspace $Z \subset X$ containing Y is lush.

The aim of this section is to show that for subspaces of real $C(K)$ spaces with K perfect, lush-wealth is equivalent to C-richness. The proof of this result uses Proposition 2.3 which is only valid for real Banach spaces.

Theorem 6.2. *Let K be a perfect compact space and let Y be a subspace of the real space $C(K)$. Then, Y is C-rich if and only if Y is lush-wealthy.*

We need the following preliminary result. We write $S^1 = \{(\alpha,\beta) \in \mathbb{R}^2 \,:\, \alpha^2+\beta^2=1\}$.

Lemma 6.3. *Let K be a perfect compact space and let V be an open subset.*

(a) *There is $\phi : K \longrightarrow S^1$ surjective and continuous such that $\phi(K\setminus V)=\{(1,0)\}$.*
(b) *Therefore, if we write $\phi(t)=(f(t),g(t))$ for every $t\in K$, then $f,g\in C(K)$ satisfy*
  (b1) $\|(\alpha f+\beta g)|_V\| = \|\alpha f+\beta g\| = (\alpha^2+\beta^2)^{1/2}$ *for every* $\alpha,\beta\in\mathbb{R}$.
  (b2) $\|(\alpha f+\beta g+c\mathbf{1})|_V\| = (\alpha^2+\beta^2)^{1/2}+|c|$ *for every* $\alpha,\beta,c\in\mathbb{R}$.
  (b3) $g|_{K\setminus V}\equiv 0$ *and* $f|_{K\setminus V}\equiv 1$.

Proof. (a). If V is clopen in K, then it is a perfect compact space and so there is an onto continuous function $\psi : V \longrightarrow S^1$ (see [17, Theorem 8.5.4]) and we may define ϕ to be $\phi|_V=\psi$ and $\phi|_{K\setminus V}=(1,0)$. Otherwise, V is not closed and so the quotient space $\widetilde{K}$ of K obtained by identifying all points of $K\setminus V$ is perfect and we may find $\psi:\widetilde{K}\longrightarrow S^1$ onto and continuous and, rotating if needed, we may suppose that $K\setminus V$ is mapped to the point $(1,0)$. Now, ϕ is just the composition of the quotient map $K\longrightarrow\widetilde{K}$ and ψ.

(b). Since $\phi(\overline{V})=S^1$ and $[\alpha f+\beta g](t) = \big((\alpha,\beta)\mid(f(t),g(t))\big)$ for every $t\in K$ and every $\alpha,\beta\in\mathbb{R}$, assertions (b1) and (b2) follow easily. The fulfilment of (b3) is straightforward. □

Proof of Theorem 6.2. If Y is C-rich, then evidently every superspace Z of Y is also C-rich and so lush by [4, Theorem 2.4].

Conversely, we assume that Y is lush-wealthy and we start with the following claim.

Claim: Given $\varepsilon>0$ and an open set $V\subset K$, we can find $c\in\mathbb{R}$ and $y\in Y$ so that

$$|y(t)+c|\leqslant\varepsilon \quad (t\in K\setminus V) \qquad\text{and}\qquad \|(y+c\mathbf{1})|_V]\|\geqslant 1+\varepsilon.$$

Proof of the Claim. We use Lemma 6.3 to get $f,g\in C(K)$ fulfilling (b1), (b2) and (b3), and we consider the space $X=\operatorname{lin}\{Y,f,g\}$ which is lush by hypothesis. Therefore, fixed $\delta>0$ such that

$$0<\delta<\tfrac{1}{1000} \qquad\text{and}\qquad 100(1+\varepsilon)\delta<\varepsilon, \tag{4}$$

we may use Proposition 2.3 to get that there exist $h\in S_X$ and $\gamma_1,\gamma_2\in\mathbb{R}$ such that

$$\|h+g\|>2-\delta^2,\quad \|f+\gamma_1 h\|<1+\delta^2,\quad \|f+\gamma_2 h\|<1+\delta^2,\ \text{and}\quad |\gamma_1-\gamma_2|=2. \tag{5}$$

Since $g|_{K\setminus V}\equiv 0$, there is $t_0\in V$ satisfying $|g(t_0)+h(t_0)|>2-\delta^2$ and so

$$|g(t_0)|>1-\delta^2 \qquad\text{and}\qquad |h(t_0)|>1-\delta^2.$$

Next, we observe that (b1) gives us that $\|g\pm\delta f\|=(1+\delta^2)^{1/2}$ and, therefore,

$$(1+\delta^2)^{1/2}\geqslant|g(t_0)|+\delta|f(t_0)|\geqslant(1-\delta^2)+\delta|f(t_0)|$$

which obviously implies

$$|f(t_0)|\leqslant\frac{(1+\delta^2)^{1/2}-(1-\delta^2)}{\delta}<2\delta.$$

This, together with (5), tells us that

$$1+\delta^2>|h(t_0)|\,|\gamma_i|-|f(t_0)|>(1-\delta^2)|\gamma_i|-2\delta \qquad (i=1,2)$$

so we obtain $|\gamma_i| < \frac{1+\delta}{1-\delta} < 1+3\delta$ for $i=1,2$ and, using the fact that $|\gamma_1-\gamma_2|=2$, we deduce that

$$|\gamma_1|, |\gamma_2| \in [1-3\delta, 1+3\delta] \qquad \text{and} \qquad \gamma_1\gamma_2 < 0.$$

Using this and (5) again, we get

$$\|f-h\| \leqslant 1+\delta^2+3\delta \leqslant 1+4\delta \qquad \text{and} \qquad \|f+h\| \leqslant 1+4\delta. \tag{6}$$

Therefore, given $t \in K \setminus V$, we have that

$$1+|h(t)| = |f(t)|+|h(t)| \leqslant 1+4\delta$$

which implies $|h(t)| \leqslant 4\delta$. Since $h \in X$, there exist $\alpha, \beta \in \mathbb{R}$ and $x \in Y$ satisfying $h = x+\alpha f+\beta g$, and thus

$$|x(t)+\alpha| \leqslant 4\delta \qquad (t \in K \setminus V).$$

Henceforth, if we show that

$$\|(x+\alpha\mathbf{1})|_V\| > \tfrac{1}{25} \tag{7}$$

the proof of the claim will be finished by just taking

$$y = 25(1+\varepsilon)x \qquad \text{and} \qquad c = 25(1+\varepsilon)\alpha.$$

In order to prove (7), on the one hand we observe that, by (b2), we have

$$(\alpha^2+\beta^2)^{1/2} \leqslant \big\|\big(\alpha f+\beta g-\alpha\mathbf{1}\big)|_V\big\| \leqslant \|h|_V\| + \|(x+\alpha\mathbf{1})|_V\| = 1+\|(x+\alpha\mathbf{1})|_V\| \tag{8}$$

and

$$1 = \|h|_V\| \leqslant \|(x+\alpha\mathbf{1})|_V\| + \|\alpha f+\beta g-\alpha\mathbf{1}\| \leqslant \|(x+\alpha\mathbf{1})|_V\| + 2(\alpha^2+\beta^2)^{1/2}. \tag{9}$$

We are now going to estimate the distance between $\|(x+\alpha\mathbf{1})|_V\|$ and $(\alpha^2+\beta^2)^{1/2}$. To do so, we call

$$G_1 = (1+\alpha)f+\beta g-\alpha\mathbf{1} \qquad \text{and} \qquad G_2 = (1-\alpha)f-\beta g+\alpha\mathbf{1},$$

we observe that

$$f+h = G_1+x+\alpha\mathbf{1} \qquad \text{and} \qquad f-h = G_2-(x+\alpha\mathbf{1}),$$

and we use (6) to get

$$\begin{aligned} 2(1+4\delta)^2 &\geqslant \|(f+h)|_V\|^2 + \|(f-h)|_V\|^2 \\ &\geqslant \big|\|(x+\alpha\mathbf{1})|_V\| - \|G_1|_V\|\big|^2 + \big|\|(x+\alpha\mathbf{1})|_V\| - \|G_2|_V\|\big|^2 \\ &= 2\|(x+\alpha\mathbf{1})|_V\|^2 - 2\|(x+\alpha\mathbf{1})|_V\|(\|G_1|_V\|+\|G_2|_V\|) + \|G_1|_V\|^2 + \|G_2|_V\|^2. \end{aligned}$$

Besides, using conditions (b1) and (b2) it is easy to show that

$$\|G_1|_V\| \leqslant 1+|\alpha|+(\alpha^2+\beta^2)^{1/2} \leqslant 1+2(\alpha^2+\beta^2)^{1/2},$$

$$\|G_2|_V\| \leqslant 1+2(\alpha^2+\beta^2)^{1/2}, \quad \text{and} \quad \|G_1|_V\|^2+\|G_2|_V\|^2 \geqslant 2+2(\alpha^2+\beta^2).$$

Hence, we deduce that

$$(1+4\delta)^2 \geqslant \|(x+\alpha\mathbf{1})|_V\|^2 - 2\|(x+\alpha\mathbf{1})|_V\|(1+2(\alpha^2+\beta^2)^{1/2}) + 1+\alpha^2+\beta^2$$

and, therefore,

$$\big(\|(x+\alpha\mathbf{1})|_V\| - (\alpha^2+\beta^2)^{1/2}\big)^2 \leqslant 2\|(x+\alpha\mathbf{1})|_V\|(1+(\alpha^2+\beta^2)^{1/2}) + 8\delta+16\delta^2$$

which, together with (8), implies

$$\big(\|(x+\alpha\mathbf{1})|_V\| - (\alpha^2+\beta^2)^{1/2}\big)^2 \leqslant 4\|(x+\alpha\mathbf{1})|_V\| + 2\|(x+\alpha\mathbf{1})|_V\|^2 + 8\delta+16\delta^2.$$

Summarizing, if we write $A = \|(x+\alpha\mathbf{1})|_V\|$ and $B = (\alpha^2+\beta^2)^{1/2}$, we deduce from (4), (9) and the above equation that

$$\begin{cases} 1 \leqslant A + 2B \\ (A-B)^2 \leqslant 4A + 2A^2 + 8\delta + 16\delta^2 \end{cases}$$

from where it is easy to deduce that $A = \|(x+\alpha\mathbf{1})|_V\| > \frac{1}{25}$, finishing the proof of the claim.

To finish the proof, we fix $\varepsilon > 0$ and an open set $U \subset K$, and we observe that it is enough to find a function $y \in S_Y$ with $\|y|_U\| = 1$ and with modulus smaller than ε on $K \setminus U$. To find this function, we consider two open subsets V_1, V_2 contained in U such that $\overline{V_1} \cap \overline{V_2} = \emptyset$ and we use the claim to find $c_1, c_2 \in \mathbb{R}$ and $y_1, y_2 \in Y$ satisfying

$$\|(y_i + c_i\mathbf{1})|_{K\setminus V_i}\| \leqslant \varepsilon/2 \quad \text{and} \quad \|(y_i + c_i\mathbf{1})|_{V_i}]\| \geqslant 1 + \varepsilon/2 \qquad (i = 1,2).$$

If either $c_1 = 0$ or $c_2 = 0$, then $\frac{y_1}{\|y_1\|}$ or $\frac{y_2}{\|y_2\|}$ is the function we are looking for. Therefore, we may assume without loss of generality that $0 < |c_1| \leqslant |c_2|$ and we take the function $\widetilde{y} = y_1 - \frac{c_1}{c_2}\, y_2 \in Y$. Now, we have

$$|\widetilde{y}(t)| \leqslant |y_1(t) + c_1| + \frac{|c_1|}{|c_2|}|y_2(t) + c_2| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (t \in K \setminus U)$$

and

$$\|\widetilde{y}|_{V_1}\| \geqslant \|(y_1 + c_1\mathbf{1})|_{V_1}\| - \frac{|c_1|}{|c_2|}\|(y_2 + c_2\mathbf{1})|_{V_1}\| \geqslant 1 + \varepsilon/2 - \varepsilon/2 = 1.$$

Therefore, $\|\widetilde{y}\| = \|\widetilde{y}|_{V_1}\| \geqslant 1$ and so the function $y = \frac{\widetilde{y}}{\|\widetilde{y}\|} \in S_Y$ satisfies

$$\|y|_{K\setminus U}\| \leqslant \|\widetilde{y}|_{K\setminus U}\| \leqslant \varepsilon \quad \text{and} \quad \|y|_U\| = 1.$$

□

We would like to remark that, outside of this result, lush-wealth differs strongly from wealth. For example, one can show that no one-codimensional subspace of $L_1[0,1]$ is lush (this is not trivial!), and so $L_1[0,1]$ does not have at all any proper lush-wealthy subspace.

Acknowledgement. The authors thank the anonymous referee for several stylistical improvements.

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