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CONVEXITY AND SMOOTHNESS OF BANACH SPACES WITH NUMERICAL INDEX ONE

VLADIMIR KADETS, MIGUEL MARTÍN, JAVIER MERÍ, AND RAFAEL PAYÁ

ABSTRACT. We show that a Banach space with numerical index one cannot enjoy good convexity or smoothness properties unless it is one-dimensional. For instance, it has no WLUR points in its unit ball, its norm is not Fréchet smooth and its dual norm is neither smooth nor strictly convex. Actually, these results also hold if the space has the (strictly weaker) alternative Daugavet property. We construct a (non-complete) strictly convex predual of an infinite-dimensional L_1 space (which satisfies a property called lushness which implies numerical index 1). On the other hand, we show that a lush real Banach space is neither strictly convex nor smooth, unless it is one-dimensional. Therefore, a rich subspace of the real space $C[0,1]$ is neither strictly convex nor smooth. In particular, if a subspace X of the real space $C[0,1]$ is smooth or strictly convex, then $C[0,1]/X$ contains a copy of $C[0,1]$. Finally, we prove that the dual of any lush infinite-dimensional real space contains a copy of ℓ_1.

1. INTRODUCTION

The classical formula $\|T\| = \sup\{|\langle Tx, x\rangle| \,:\, x \in X,\ \|x\| = 1\}$ for the norm of a self-adjoint operator T on a Hilbert space X can be rewritten, thanks to the well-known representation of the dual X^* as

$$\|T\| = \sup\{|x^*(Tx)| \,:\, x \in X,\ x^* \in X^*,\ x^*(x) = \|x^*\| = \|x\| = 1\}. \tag{1}$$

For a non self-adjoint operator this formula may fail. Nevertheless, there are some Banach spaces X in which equality (1) is valid for *every* bounded linear operator T on X. As we will explain below, such spaces are said to have numerical index 1. Among these spaces are all classical $C(K)$ and $L_1(\mu)$ spaces.

Given a real or complex Banach space X, we write B_X, S_X, X^* and $L(X)$, to denote, respectively, the closed unit ball, the unit sphere, the topological dual and the Banach algebra of bounded linear operators on X.

The *numerical range* of an operator $T \in L(X)$ is the subset of the base field given by

$$V(T) = \{x^*(Tx) \,:\, x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\},$$

and the *numerical radius* of T is then given by $v(T) = \sup\{|\lambda| \,:\, \lambda \in V(T)\}$. These concepts were independently introduced by F. Bauer [3] and G. Lumer [31] in the 1960's to extend the classical field of values of matrices (O. Toeplitz, 1918 [43]). We refer the reader to the monographs by F. Bonsall

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and J. Duncan [4, 5] for a detailed account. The *numerical index* of the space X (Lumer, 1968 [15]) is the constant $n(X)$ defined by

$$n(X) := \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\}$$

or, equivalently, the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. Observe that $0 \leqslant n(X) \leqslant 1$ for every Banach space X, and $n(X) = 1$ if and only if equality (1) is valid for all operators on X. The reader will find the state-of-the-art on numerical indices in the recent survey paper [20] to which we refer for background.

Let us mention here several facts concerning the numerical index which are relevant to our discussion. Examples of Banach spaces having numerical index 1 are $C(K)$ spaces, $L_1(\mu)$ spaces, Lindenstrauss spaces (i.e. isometric preduals of $L_1(\mu)$ spaces) [15], all function algebras [44] (for instance, the disk algebra $A(\mathbb{D})$ and H^∞), and finite-codimensional subspaces of $C[0,1]$ [8]. Next, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [4, §9]), and it clearly follows that $n(X^*) \leqslant n(X)$ for every Banach space X. It has recently been discovered that this inequality can be strict. Actually, in [8, Example 3.1] an example is given of a real Banach space X such that $n(X) = 1$ while $n(X^*) = 0$. We refer to the very recent paper [34] for sufficient conditions to ensure the equality in the inequality $n(X^*) \leqslant n(X)$. Every separable Banach space containing c_0 can be equivalently renormed to have numerical index 1 [7, §4], in particular, this happens with any closed subspace of c_0. On the other hand, there is no infinite-dimensional real reflexive space with numerical index 1 [30].

Our main goal in this paper is to study which convexity or smoothness properties are possible for the unit ball of a Banach space with numerical index 1. At the end of this introduction we give the necessary definitions of the convexity and smoothness properties we use along the paper. A difficulty with such a study is that the property of having numerical index 1 deals with all operators on the space and we do not know of any characterization of it in terms of the space and its successive duals. The previous solutions to this difficulty have been to deal with either weaker or stronger geometrical properties. Let us briefly give an account of some of them. Let X be a real or complex Banach space.

- (a) X is said to be a *CL-space* if B_X is the absolutely convex hull of every maximal convex subset of S_X.
- (b) We say that X is an *almost-CL-space* if B_X is the closed absolutely convex hull of every maximal convex subset of S_X.
- (c) X is *lush* if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice

  $$S = S(x^*, \varepsilon) := \{z \in B_X \ : \ \operatorname{Re} x^*(z) > 1 - \varepsilon\}$$

  with $x^* \in S_{X^*}$ such that $x \in S$ and $\operatorname{dist}(y, \operatorname{aco}(S)) < \varepsilon$, where $\operatorname{aco}(S)$ denotes the absolutely convex hull of the set S.
- (d) X has *numerical index* 1 ($n(X) = 1$ in short) if $v(T) = \|T\|$ for every $T \in L(X)$.
- (e) We say that X has the *alternative Daugavet property* provided that every rank-one operator $T \in L(X)$ satisfies $v(T) = \|T\|$. The same equality is then satisfied by all weakly compact operators on X [36, Theorem 2.2].

The implications $(a) \Longrightarrow (b) \Longrightarrow (c)$ and $(d) \Longrightarrow (e)$ are clear and none of them reverses (see [8, §3 and §7] for a detailed account). Also, $(c) \Longrightarrow (d)$ by [8, Proposition 2.2].

Some additional comments on the above properties may be in place. CL-spaces where introduced in 1960 by R. Fullerton [17] and it was later shown that a finite-dimensional Banach space has numerical index 1 if and only if it is a CL-space ([38, Theorem 3.1] and [27, Corollary 3.7]). Therefore, the above five properties are equivalent in the finite-dimensional case. All $C(K)$ spaces as well as real $L_1(\mu)$ spaces are CL-spaces, while infinite-dimensional complex $L_1(\mu)$ spaces are only almost-CL-spaces (see [37]). Almost-CL-spaces first appeared without a name in the memoir by J. Lindenstrauss [28] and were further discussed by Å. Lima [26, 27] who showed that real Lindenstrauss spaces (i.e. isometric preduals of $L_1(\mu)$) are CL-spaces [26, §3] and complex Lindenstrauss spaces are almost-CL-spaces [27, §3]. The disk algebra is another classical example of an almost-CL-space [5, Theorem 32.9]. More information can be found in [9, 32, 37, 41].

Lush spaces were introduced recently [8] and they were the key to provide an example of a Banach space X such that $n(X^*) < n(X)$ and to estimate the polynomial numerical index of some spaces [10, 23]. We refer to [7] for characterizations and examples of lush spaces. Among the advantages of the concept of lushness are that this property is separably determined [7, Theorem 4.2] and that it gives many new examples of Banach spaces with numerical index 1. Namely, C-rich subspaces of $C(K)$ are lush and so they have numerical index 1 [8, Theorem 2.4]. A closed subspace X of a $C(K)$ space is said to be *C-rich* if for every nonempty open subset U of K and every $\varepsilon > 0$, there is a positive function $h \in C(K)$ of norm 1 with support inside U such that the distance from h to X is less than ε. This definition covers finite-codimensional subspaces of $C[0,1]$ [8, Proposition 2.5], so they are lush. Also, all function algebras are lush (see [7, Example 2.4] and [44, §3]).

The alternative Daugavet property was introduced and characterized in [36] but, in an equivalent way, the property defining it had appeared in some papers of the 1990's. The name comes from the fact that an operator T on a Banach space X satisfies $v(T) = \|T\|$ if and only if $\|\mathrm{Id} + \omega\, T\| = 1 + \|T\|$ for some $\omega \in \mathbb{T}$ ($\mathbb{T}$ being the set of modulus one scalars) [15], that is, the operator $S = \omega\, T$ satisfies the so-called Daugavet equation $\|\mathrm{Id} + S\| = 1 + \|S\|$. Therefore, X has the alternative Daugavet property if and only if every rank-one operator (equivalently, every weakly compact operator) satisfies the Daugavet equation up to rotation. We refer to the already cited paper [36] and to [33] for more information and background. Let us comment that Banach spaces with the Radon-Nikodým property and the alternative Daugavet property are actually almost-CL-spaces [32]. Asplund spaces with the alternative Daugavet property are lush, but they need not be almost-CL-spaces [8, Example 2.4].

The main question in this paper, not yet solved, is whether a Banach space X with $n(X) = 1$ can be smooth or strictly convex. Two remarks are pertinent. First, even though the exact value of $n(\ell_p^2)$ is not known for $p \neq 1, 2, \infty$ (see [35]), it is known that the set

$$\left\{ n\left(\ell_p^2\right) \,:\, 1 < p < \infty \right\}$$

contains all possible values of the numerical index except 1 [15]. Thus, the question above only makes sense for the value 1. Second, it is clear that an almost-CL-space cannot be strictly convex (almost-CL-spaces are somehow the extremely opposite property to strict convexity), and it has recently been shown that a real almost-CL-space cannot be smooth [9, Theorem 3.1].

Let us summarize the main results in this paper. Section 2 is devoted to show that a Banach space with the alternative Daugavet property and dimension greater than one has no WLUR points in its unit ball, its norm is not Fréchet smooth and its dual norm is neither strictly convex nor smooth. Next, in §3 we construct a non-complete predual of an $L_1(\mu)$ space which is strictly convex. This space is lush (extending this definition to general normed spaces literally), while its completion is an almost-CL-space. The aim of section 4 is to show that separable lush spaces actually satisfy a stronger property: there is a norming subset $\tilde{K}$ of S_{X^*} such that for every $x^* \in \tilde{K}$ and every $\varepsilon > 0$, one has

$$B_X = \overline{\mathrm{aco}\big(S(x^*, \varepsilon)\big)}.$$

In the real case, it is actually true that B_X is the closed absolutely convex hull of the (non-empty) face generated by x^*. This implies that a real lush Banach space is neither strictly convex nor smooth, unless it is one-dimensional. Therefore, a C-rich subspace of the real space $C[0,1]$ is neither strictly convex nor smooth, and this answers a question of M. Popov from 1996. In particular, if a subspace X of the real space $C[0,1]$ is smooth or strictly convex, then $C[0,1]/X$ contains a copy of $C[0,1]$. We devote §5 to some localizations of convexity and smoothness properties. Namely, it was asked in [20, Problem 13] whether a Banach space X with $n(X) = 1$ satisfies that $|x^*(x)| = 1$ for every $x \in \operatorname{ext}(B_X)$ (the set of extreme points in B_X) and every $x^* \in \operatorname{ext}(B_{X^*})$, as it happens in the finite-dimensional case [38] (a positive answer would lead to the impossibility of having a strictly convex space with numerical index 1 other than the one-dimensional one). But actually, we construct examples of separable lush spaces where this does not happen, giving a negative answer to the cited problem. On the other hand, we show that for lush spaces, $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$ and every w^*-extreme point, which gives us that a lush space which is WMLUR has to be one-dimensional.

We finish the introduction with the definitions and notations of the convexity and smoothness properties that we need throughout the paper. We refer the reader to the books [11, 12] and the papers [2, 24] for more information and background.

The norm of a real or complex Banach space X (or X itself) is said to be *smooth* if for every $x \in S_X$, there is a unique norm-one functional x^* such that $x^*(x) = 1$. The space X is said to be *strictly convex* when $\operatorname{ext}(B_X) = S_X$. It is well known that X is smooth (resp. strictly convex) if X^* is strictly convex (resp. smooth), but the converse is not true. We say that the norm of X is *Fréchet smooth* when the norm of X is Fréchet differentiable at any point of S_X. By the Smulyan test, the norm of X is Fréchet smooth if and only if every functional $x^* \in S_{X^*}$ which attains its norm is w^*-strongly exposed (i.e. there is $x \in S_X$ such that for every sequence (x_n^*) in B_{X^*} such that $x_n^*(x) \longrightarrow 1 = x^*(x)$ one has $x_n^* \longrightarrow x^*$ in norm).

An $x \in S_X$ is said to be a point of *local uniform rotundity (LUR point)* if $\|x_n - x\| \longrightarrow 0$ for every sequence (x_n) in S_X such that $\|x_n + x\| \longrightarrow 2$. If for every sequence (x_n) of S_X with $\|x_n + x\| \longrightarrow 2$ one only has that $x_n \longrightarrow x$ in the weak topology, we say that x is a point of *weakly local uniform rotundity (WLUR point)*.

A point x in S_X is said to be (*weakly*) *midpoint locally uniformly rotund* or *MLUR* (resp. *WMLUR*) if for any sequence (y_n) in B_X, $\lim_n \|x \pm y_n\| \leqslant 1$ implies $\lim_n \|y_n\| = 0$ ($\lim_n y_n = 0$ in the weak topology). A point x of B_X is called *weak*-extreme* if it is an extreme point of $B_{X^{**}}$. Every WMLUR point of B_X is a weak*-extreme point of B_X (see [18, p. 674] and [24, p. 173]). We say that the norm of X is MLUR (WMLUR) if every point in S_X is MLUR (WMLUR).

2. Prohibitive results for the alternative Daugavet property

The aim in this section is to show that there are some convexity and smoothness properties which are incompatible with the alternative Daugavet property and so, they are incompatible with the numerical index 1. We start with smoothness and strict convexity of the dual norm.

Theorem 2.1. *Let X be a Banach space with the alternative Daugavet property and dimension greater than one. Then, X^* is neither smooth nor strictly convex.*

Proof. Since the dimension of X is greater than 1, we may find $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 0$. Then, we consider the norm-one operator $T = x_0^* \otimes x_0$, which satisfies $T^2 = 0$. On the other hand, thanks to [1, Theorem 1.2], there is a sequence of norm-one operators (T_n) converging in norm to T and such that the adjoint of each of them attains its numerical radius. Moreover, we may suppose that all the T_n's are compact by [1, Remark 1.3]. Since X has the alternative Daugavet property, we get

$$v(T_n^*) = v(T_n) = \|T_n\| = 1.$$

As the operators T_n^* attain their numerical radius, for every positive integer n, we may find $\lambda_n \in \mathbb{T}$ and $(x_n^*, x_n^{**}) \in S_{X^*} \times S_{X^{**}}$ such that

$$\lambda_n\, x_n^{**}(x_n^*) = 1 \qquad \text{and} \qquad \big[T_n^{**}(x_n^{**})\big](x_n^*) = x_n^{**}(T_n^*(x_n^*)) = 1. \tag{2}$$

If X^* is smooth, we deduce that

$$T_n^{**}(x_n^{**}) = \lambda_n\, x_n^{**} \qquad \big(n \in \mathbb{N}\big).$$

Thus,

$$\Big\| \left[T_n^{**}\right]^2 (x_n^{**}) \Big\| = \|\lambda_n^2\, x_n^{**}\| = 1 \qquad (n \in \mathbb{N}).$$

But, since $T_n \longrightarrow T$ and $T^2 = 0$, we have that $\left[T_n^{**}\right]^2 \longrightarrow 0$, a contradiction.

If X^* is strictly convex, we deduce from (2) that

$$T_n^*(x_n^*) = \lambda_n\, x_n^* \qquad \big(n \in \mathbb{N}\big),$$

which leads to a contradiction the same way as before. □

As a consequence of the above result, we get that $n(H^1) < 1$, where H^1 represents the Hardy space. Actually, we have more.

Example 2.2. *Let X be $C(\mathbb{T})/A(\mathbb{D})$. Then, its dual $X^* = H^1$ is smooth (see* [19, Remark IV.1.17], *for instance), so X does not have the alternative Daugavet property by Theorem 2.1 and neither does $X^* = H^1$. In particular, $n(X) < 1$ and $n(X^*) < 1$.*

Remarks 2.3.

(a) The proof of Theorem 2.1 can be adapted to yield the following result. *Let X be a Banach space with the alternative Daugavet property and such that the set of compact operators attaining its numerical radius is dense in the space of all compact operators. Then, X is neither strictly convex nor smooth, unless it is one-dimensional.* Indeed, we may follow the proof of Theorem 2.1 (without considering adjoint operators) to get the result.

(b) It is known that for Banach spaces with the Radon-Nikodým property, the set of compact operators attaining their numerical radius is dense in the space of all compact operators [1, Theorem 2.4]. Therefore, we get that *a Banach space having the Radon-Nikodým property and the alternative Daugavet property is neither smooth nor strictly convex, unless it is one-dimensional.*

(c) Actually, the above result was essentially known. Namely, if X has the alternative Daugavet property and the Radon-Nikodým property, then X is an almost-CL-space [32, Theorem 1]. It is clear that a (non-trivial) almost-CL-space cannot be strictly convex. On the other hand, the fact that a non-trivial real almost-CL-space cannot be smooth follows from a very recent result [9, Theorem 3.1].

(d) The fact that there are Banach spaces in which the set of numerical radius attaining operators is not dense in the space of all operators was discovered in 1992 [39]. Nevertheless, *we do not know of any Banach space for which the set of compact operators which attain their numerical radius is not dense in the space of all compact operators.*

(e) Let us comment that it is also an open problem whether a Banach space with the Daugavet property can be smooth or strictly convex. We recall that a Banach space has the *Daugavet property* if $\|\mathrm{Id} + T\| = 1 + \|T\|$ for every rank-one operator $T \in L(X)$ [22]. It is clear that the Daugavet property implies the alternative Daugavet property (and the converse result is not true). Therefore, an example of a smooth or strictly convex Banach space with the Daugavet property would give an example of a Banach space where the rank-one operators cannot be approximated by compact operators attaining the numerical radius.

More prohibitive results for the alternative Daugavet property are the following.

Proposition 2.4. *Let X be a Banach space with the alternative Daugavet property. Then, B_X fails to contain a WLUR point, unless X is one-dimensional.*

Proof. Let $x_0 \in S_X$ be a WLUR point. If the dimension of X is greater than 1, there is $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 0$. Then, the rank-one operator $T = x_0^* \otimes x_0$ satisfies $\|T\| = 1$ and so, $v(T) = 1$. Therefore, we may find sequences (x_n) in S_X and (x_n^*) in S_{X^*} such that

$$x_n^*(x_n) = 1 \qquad \text{and} \qquad |x_n^*(x_0)|\,|x_0^*(x_n)| = |x_n^*(Tx_n)| \longrightarrow 1.$$

Therefore, we get $|x_n^*(x_0)| \longrightarrow 1$ and $|x_0^*(x_n)| \longrightarrow 1$. If for every $n \in \mathbb{N}$ we take $\lambda_n \in \mathbb{T}$ such that $x_n^*(x_0) = \lambda_n\,|x_n^*(x_0)|$, we have

$$2 \geqslant \|x_0 + \lambda_n\, x_n\| \geqslant |x_n^*(x_0 + \lambda_n\, x_n)| \longrightarrow 2.$$

So, being x_0 a WLUR point, we get that $(\lambda_n\, x_n) \longrightarrow x_0$ in the weak topology, which contradicts the fact that $|x_0^*(x_n)| \longrightarrow 1$. ☐

The above result is not true if we replace the WLUR point by a point of Fréchet smoothness. For instance, $n(c_0) = 1$ but the norm of c_0 is Fréchet differentiable at a dense subset of S_{c_0} since c_0 is Asplund. But it is not difficult to show that a Banach space with the alternative Daugavet property cannot have a Fréchet smooth norm, unless it is one-dimensional.

Proposition 2.5. *Let X be a Banach space with the alternative Daugavet property. Then, the norm of X is not Fréchet smooth, unless X is one-dimensional.*

Proof. Using [30, Lemma 1], we have that $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every w^*-strongly exposed point x^* of B_{X^*}. Now, if the norm of X is Fréchet-smooth, then every functional on S_{X^*} attaining its norm is w^*-strongly exposed (see [11, Corollary I.1.5] for instance). Since, by the Bishop-Phelps Theorem, we have that the set of norm-attaining norm-one functionals is (norm) dense on S_{X^*}, we get that

$$|x^{**}(x^*)| = 1$$

for all $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and all $x^* \in S_{X^*}$. This clearly leads to the fact that X is one-dimensional. ☐

3. A NONCOMPLETE STRICTLY CONVEX LUSH SPACE

The aim of this section is to construct an example of a non-complete infinite-dimensional strictly convex normed space with numerical index 1 (actually lush). As we will see, its completion is very far away from being strictly convex. In the next section, we will show that actually no real lush complete space can be strictly convex.

We need some definitions and preliminary results.

Definitions 3.1. Let $|||\cdot|||$ and $\|\cdot\|$ be two norms on a linear space X and $\varepsilon > 0$. We say that $|||\cdot|||$ is *ε-equivalent* to $\|\cdot\|$ if

$$\frac{1}{1+\varepsilon}\|x\| \leqslant |||x||| \leqslant (1+\varepsilon)\|x\| \qquad (x \in X).$$

A property $\mathcal{P}$ of normed spaces is said to be a *stable C-property*, if $C[0,1] \in \mathcal{P}$ and for every Banach space X the following condition is sufficient for $X \in \mathcal{P}$:
for every $\varepsilon > 0$ and for every finite subset $F \subset X$, there is a subspace $Y \subset X$, such that $F \subset Y$ and Y possesses an ε-equivalent norm $\|\cdot\|_\varepsilon$ with $(Y, \|\cdot\|_\varepsilon) \in \mathcal{P}$.

It is immediate that lushness and the alternative Daugavet property are stable C-properties.

We are now ready to state the main result to get the example.

Theorem 3.2. *For every strictly convex separable Banach space Y_0, there is a strictly convex separable normed space $X \supset Y_0$ possessing all stable C-properties.*

We will need the following surely well-known lemma.

Lemma 3.3. *Let Y be a strictly convex closed subspace of a separable Banach space X. Then for every $\varepsilon > 0$, there is an ε-equivalent strictly convex norm $|||\cdot|||$ on X which coincides with the original one on Y.*

Proof. The existence of a norm p satisfying all conditions of this statement except being ε-equivalent to the original one is well known (see for example [11, p. 84] or [42, Theorem 1.1]). Then, for sufficiently small $t > 0$, the norm $|||x||| := (1-t)\|x\| + tp(x)$ will be the one which we need. □

Proof of Theorem 3.2. We are going to construct a sequence of separable strictly convex Banach spaces (X_n) with the following properties:

(i) $X_1 = Y_0$.
(ii) X_n is a subspace of X_m for $n < m$.
(iii) For every $n \in \mathbb{N}$ there is a $\frac{1}{n}$-equivalent norm $\|\cdot\|_n$ on X_n with $(X_n, \|\cdot\|_n)$ being isometric to $C[0,1]$.

Since X_1 is already known, the only thing we need for this construction is to show how to get X_{m+1} from X_m. Let us fix $m \in \mathbb{N}$. Since X_m is separable, we can (and do so) consider X_m as a subspace of $C[0,1]$. According to Lemma 3.3, there is an $(\frac{1}{m})$-equivalent strictly convex norm $|||\cdot|||$ on $C[0,1]$ which coincides with the original norm on X_m. Put $X_{m+1} = \big(C[0,1], |||\cdot|||\big)$, and the original norm of $C[0,1]$ plays the role of $\|\cdot\|_n$ in the condition (iii). So the construction is completed.

What remains to complete the proof itself is to put $X = \bigcup_{m\in\mathbb{N}} X_m$. Then, for every $\varepsilon > 0$ and for every finite subset $F \subset X$, one can find $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and $F \subset X_n$. Since $\|\cdot\|_n$ is ε-equivalent to the norm of X_n, we get the requirement. □

Since lushness is a stable C-property, we get the desired example.

Example 3.4. There are normed lush spaces which are strictly convex.

We are going to show that the completions of the above examples (which are of course also lush) are not strictly convex. Actually, they are almost-CL-spaces.

Following Bourgain's book [6], we say that a Banach space X is an $\mathcal{L}^{\infty}_{1+}$*-space* if for any finite-dimensional subspace E of X and every $\varepsilon > 0$, there is another finite-dimensional subspace F of X containing E such that the Banach-Mazur distance between F and $\ell_\infty^{(\dim(F))}$ is less than $1+\varepsilon$. It is well known [25] that this property is equivalent to the fact that X^* is isometrically isomorphic to an $L_1(\mu)$ space. The completions of the spaces constructed in Theorem 3.2 are $\mathcal{L}^{\infty}_{1+}$-spaces, so they are preduals of $L_1(\mu)$ spaces. In the real case, it is known that preduals of $L_1(\mu)$ spaces are almost-CL-spaces (see [28, Theorem 4.8] or [27, Corollary 3.6]). Actually, the same is true for the complex case. We include here a proof of this fact since we have been unable to find it in the literature.

Proposition 3.5. *Let X be a (real or complex) Banach space such that X^* is isometrically isomorphic to an $L_1(\mu)$ space. Then, X is an almost-CL-space.*

Proof. If we consider a maximal convex subset F of B_X, the Hahn-Banach and Krein-Milman theorems ensure that there is an extreme point f of the unit ball of $X^* = L_1(\mu)$ such that

$$F = F(f) := \{x \in B_X \ : \ f(x) = 1\}.$$

We observe that the linear span of an extreme point f in the unit ball of an $L_1(\mu)$ space is an L-summand (i.e. $L_1(\mu) = \mathrm{lin}(f) \oplus_1 Z$ for some closed subspace Z). So, a result by Å. Lima [27, Theorem 5.3] says that $F(f)$ is not empty (we already knew it) and that B_X is the closure of $\mathrm{aco}(F(f))$. This shows that X is an almost-CL-space. □

As a immediate consequence of this result we get the following.

Corollary 3.6. *The completions of the non-complete lush strictly convex normed spaces constructed in Theorem 3.2 are almost-CL-spaces and, therefore, they are not strictly convex.*

We finish the section by remarking that the arguments of the construction given in Theorem 3.2 cannot be adapted for smoothness, since a smooth norm on a subspace cannot always be extended to the whole space (see [11, Theorem 8.3]).

4. Separable lush spaces

We have seen in the previous section that the completions of the normed strictly convex lush spaces constructed are not strictly convex by showing that they are almost-CL-spaces. We cannot expect that every Banach space with numerical index 1 is an almost-CL-space since there are lush spaces which do not fulfil this property [8, Example 3.4]. Nevertheless, our aim here is to show that, in the separable case, lush spaces actually have a much stronger property which in the real case is very close to being an almost-CL-space and which will allow us to show that a real lush space cannot be strictly convex, unless it is one-dimensional.

We need a characterization of lushness given in [7] in terms of a norming subset of S_{X^*}. Also, to carry some consequences to the non-separable case, we need a result of the same paper saying that lushness is a separably determined property. We state both results here for easier reference.

Proposition 4.1 ([7, Theorems 4.1 and 4.2])**.** *Let X be a Banach space.*

(a) *The following assertions are equivalent:*
  - *(i) X is lush.*
  - *(ii) For every $x, y \in S_X$ and for every $\varepsilon > 0$ there is a slice $S = S(x^*, \varepsilon) \subset B_X$, $x^* \in \mathrm{ext}(B_X)$, such that*

$$x \in S \qquad \text{and} \qquad \mathrm{dist}\,(y, \mathrm{aco}(S)) < \varepsilon$$

  (i.e. x^* in the definition of lushness can be chosen from $\mathrm{ext}(B_X)$).

(b) *The following two conditions are equivalent:*
  - *(i) X is lush,*
  - *(ii) Every separable subspace $E \subset X$ is contained in a separable lush subspace Y, $E \subset Y \subset X$.*

The following lemma, which will be the key to prove the main result of the section, will be also useful in section 5 and does not depend upon the separability of the space.

Lemma 4.2. *Let X be a lush space and let $K \subset B_{X^*}$ be the weak* closure of $\mathrm{ext}(B_{X^*})$ endowed with the weak* topology. Then, for every $y \in S_X$, there is a G_δ-dense subset K_y of K such that $y \in \overline{\mathrm{aco}(S(y^*, \varepsilon))}$ for every $\varepsilon > 0$ and every $y^* \in K_y$.*

Proof. Fix $y \in S_X$. For every $n, m \in \mathbb{N}$, we consider

$$K_{y,n,m} := \{x^* \in K \,:\, \operatorname{dist}(y, \operatorname{aco}(S(x^*, 1/n))) < 1/m\}.$$

Claim. $K_{y,n,m}$ is weak*-open and dense in K.

In fact, openness is almost evident: if $x^* \in K_{y,n,m}$, then there is a finite set $A = \{a_1, \ldots a_k\}$ of elements of $S(x^*, 1/n)$ such that $\operatorname{dist}(y, \operatorname{aco}(A)) < 1/m$. Denote

$$U := \{y^* \in K : \operatorname{Re} y^*(a_i) > 1 - 1/n \text{ for all } i = 1, \ldots, k\}.$$

U is a weak*-neighborhood of x^* in K, and $A \subset S(y^*, 1/n)$ for every $y^* \in U$. This means that $\operatorname{dist}(y, \operatorname{aco}(S(y^*, 1/n))) < 1/m$ for all $y^* \in U$, i.e. $U \subset K_{y,n,m}$.

To show density of $K_{y,n,m}$ in K, it is sufficient to demonstrate that the weak* closure of $K_{y,n,m}$ contains every extreme point x^* of S_{X^*}. Since weak*-slices form a base of neighborhoods of x^* in B_{X^*} (see [16, Lemma 3.40], for instance), it is sufficient to prove that every weak*-slice $S(x, \delta)$, $\delta \in (0, \min\{1/n, 1/m\})$, intersects $K_{y,n,m}$, i.e. that there is a point $y^* \in S(x, \delta) \cap K_{y,n,m}$. Which property of y^* do we need to make this true? We need that $y^*(x) > 1 - \delta$, $y^* \in K$, and that $\operatorname{dist}(y, \operatorname{aco}(S(y^*, 1/n))) < 1/m$. But the existence of such a y^* is a simple application of item (ii) from Proposition 4.1.(a). The claim is proved.

Now, we consider $K_y = \bigcap_{n,m\in\mathbb{N}} K_{y,n,m}$, which is a weak*-dense G_δ subset of K due to the Baire theorem. □

We are now ready to state and prove the main result of the section.

Theorem 4.3. *Let X be a separable lush space. Then, there is a norming subset $\tilde{K}$ of S_{X^*} such that $B_X = \overline{\operatorname{aco}(S(x^*, \varepsilon))}$ for every $\varepsilon > 0$ and for every $x^* \in \tilde{K}$. The last condition implies that*

$$|x^{**}(x^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{X^{**}}),\ x^* \in \tilde{K}\big),$$

and that in fact $\tilde{K} \subset \operatorname{ext}(B_{X^*})$.

Proof. We select a sequence (y_n) dense in S_X in such a way that every element of the sequence is repeated infinitely many times, and consider $\tilde{K} = \bigcap_{n\in\mathbb{N}} K_{y_n}$. Due to the Baire theorem, $\tilde{K}$ is a weak*-dense G_δ subset of K. This implies that for every $x \in S_X$ and for every $\varepsilon > 0$ there is an $x^* \in \tilde{K}$, such that $x \in S(x^*, \varepsilon)$ (i.e. $\tilde{K}$ is 1-norming). For $x_0^* \in \tilde{K}$ and $\varepsilon > 0$ fixed, the inequality $\operatorname{dist}(y_n, \operatorname{aco}(S(x_0^*, 1/n))) < 1/n$ holds true for all $n \in \mathbb{N}$. Select an $N > 1/\varepsilon$. Then, for every $n > N$ we have $\operatorname{dist}(y_n, \operatorname{aco}(S(x_0^*, \varepsilon))) < 1/n$. Since every element of the sequence (y_n) is repeated infinitely many times, this means that $\operatorname{dist}(y_n, \operatorname{aco}(S(x_0^*, \varepsilon))) = 0$. So the closure of $\operatorname{aco}(S(x_0^*, \varepsilon))$ contains the whole ball B_X. Then,

$$B_{X^{**}} = \overline{B_X}^{w^*} \subseteq \overline{\operatorname{aco}(S(x_0^*, \varepsilon))}^{w^*}.$$

Finally, the reversed Krein-Milman theorem gives us that

$$\operatorname{ext}(B_{X^{**}}) \subset \overline{\mathbb{T}S(x_0^*, \varepsilon)}^{w^*},$$

and the arbitrariness of $\varepsilon > 0$ gives us

$$|x^{**}(x_0^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{X^{**}})\big).$$ □

We do not know whether the statement of the theorem is true in the non-separable case.

Remark 4.4. From the proof of the above theorem it follows that the set $\tilde{K}$ is actually a G_δ-dense subset of the weak* closure of $\operatorname{ext}(B_{X^*})$ endowed with the weak* topology.

As a consequence of the above theorem and results of Å. Lima [27], we get the following interesting version valid in the real case.

Corollary 4.5. *Let X be a lush **real** separable space. Then, there is a subset A of S_{X^*} norming for X such that for every $a^* \in A$ one has*

$$B_X = \overline{\operatorname{aco}\big(\{x \in S_X \,:\, a^*(x) = 1\}\big)}\,.$$

Proof. By the above theorem, there is a subset A of S_{X^*} norming for X such that

$$|x^{**}(a^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{X^{**}}),\ a^* \in A\big).$$

Now, Theorems 3.1 and 3.5 of [27] give us that each $a^* \in A$ attains its norm on X and, moreover, that the closed absolutely convex hull of the points of B_X where a^* attains its norm is the whole ball B_X, as claimed. □

Corollary 4.6. *Let X be a **real** Banach space which is lush. Then, X is neither strictly convex nor smooth, unless it is one-dimensional.*

Proof. If X is a lush space, then every separable closed subspace Z of X is contained in a separable lush subspace Y by Proposition 4.1.(b), and Corollary 4.5 provides us with a face F of B_Y such that $B_Y = \overline{\operatorname{co}(F \cup -F)}$. If Y is not one-dimensional, then F contains at least two distinct points y_1, y_2 and $\frac{1}{2}(y_1 + y_2) \in F \subset S_Y$ is not extreme. On the other hand, if Y is not one-dimensional, following the proof of [9, Theorem 3.1], we get that the smooth points of F are exactly the norm-one elements of the cone generated by F which are not support points of the cone. But then, the Bishop-Phelps theorem provides us with (norm-one) support points of such a cone (see [40, Theorem 3.18] for instance). Then, F and so S_Y contains non-smooth points. Therefore, Y is not smooth, all the more X. □

We do not know whether the above two results are true in the complex case. We do not know either whether there are real strictly convex Banach spaces with numerical index 1 others than $\mathbb{R}$.

As a consequence of the corollary above, we get a negative answer to a problem by M. Popov, which he posed to the first author in 1996 while discussing the still open problem on the existence of a strictly convex Banach space with the Daugavet property.

Corollary 4.7. *A C-rich closed subspace of the **real** space $C[0,1]$ is neither strictly convex nor smooth.*

It is known that a subspace X of $C[0,1]$ is C-rich whenever $C[0,1]/X$ does not contain a copy of $C[0,1]$ (see [21, Proposition 1.2 and Definition 2.1]). Therefore, the following is a particular case of the above proposition.

Corollary 4.8. *Let X be a closed subspace of the **real** space $C[0,1]$. If X is smooth or strictly convex, then $C[0,1]/X$ contains an isomorphic copy of $C[0,1]$.*

Finally, another interesting consequence of Theorem 4.3 is the following.

Corollary 4.9. *Let X be an infinite-dimensional **real** Banach space which is lush. Then X^* contains an isomorphic copy of ℓ_1.*

Proof. If X is lush, by Proposition 4.1.(b), there is an infinite-dimensional separable closed subspace Y of X which is lush. Then, by Theorem 4.3, there is a norming subset $\tilde{K}$ of S_{Y^*} (in particular, $\tilde{K}$ is infinite) such that

$$|y^{**}(y^*)| = 1 \qquad \big(y^{**} \in \operatorname{ext}(B_{Y^{**}}),\ y^* \in \tilde{K}\big).$$

Now, Proposition 2 of [30] shows that Y^* contains either c_0 or ℓ_1. But a dual space contains ℓ_∞ (hence also ℓ_1) as soon as it contains c_0 (see [13, Theorem V.10] or [29, Proposition 2.e.8], for instance). Finally, if Y^* contains a copy of ℓ_1, then so does X^* (see [14, p. 11], for instance). $\square$

The above corollary has already been known for real spaces with numerical index 1 which are Asplund or have the Radon-Nikodým property [30], and for real almost-CL-spaces [37].

5. Extreme points of the unit ball

The fact that a finite-dimensional strictly convex Banach space with numerical index 1 has to be one-dimensional is a direct consequence of an old result by C. McGregor [38]. Namely, if X is a finite-dimensional Banach space with $n(X) = 1$, then

$$|x^*(x)| = 1 \qquad \big(x^* \in \operatorname{ext}(B_{X^*}),\ x \in \operatorname{ext}(B_X)\big).$$

In [20, Problem 13] it was asked whether the above result is also true in the infinite dimensional case. There are two goals in this section. On the one hand, we will show that this is not the case. We present two examples of lush spaces (actually, C-rich subspaces of $C(K)$) such that there are $x_0^* \in \operatorname{ext}(B_{X^*})$ and $x_0 \in \operatorname{ext}(B_X)$ with $|x_0^*(x_0)| = 0$. On the other hand, we will show that such an example is not possible when the point x_0 is actually extreme in $B_{X^{**}}$. In particular, we obtain that a lush space which is MLUR or WMLUR must be one-dimensional.

Let us start with the first two examples. We give two different constructions, one for both the real and the complex case and another one for the complex case only, showing that the answer to the already mentioned Problem 13 of [20] is negative.

Example 5.1. *There is a C-rich subspace X of the space $C[0,1]$ (hence X is lush) and there are $f_0 \in \operatorname{ext}(B_X)$ and $x_0^* \in \operatorname{ext}(B_{X^*})$ satisfying $x_0^*(f_0) = 0$.*

Proof. Let us fix a function $f_0 \in C[0,1]$ such that $\|f_0\| = 1$, $f_0(t) = 0$ for $t \in (0,1/3)$, and $f_0(t) = 1$ for $t \in (2/3,1)$. We select a sequence of intervals $\Delta_n \subset [0,1]$, $|\Delta_n| < 1/3$, such that for every $(a,b) \subset [0,1]$ there is a $\Delta_j \subset (a,b)$. Also, fix a null sequence (ε_n), $\varepsilon_n > 0$. Now one can easily construct functions $f_n \in C[0,1]$, $n \in \mathbb{N}$, and functionals $f_n^* \in C[0,1]^*$, $n = 0,1,2\ldots$, recursively with the following properties:

(i) $\|f_n\| = 1$, $\|f_n|_{\Delta_n}\|_\infty = 1$ and $\|f_n|_{[0,1]\setminus\Delta_n}\|_\infty \leqslant \varepsilon_n$.
(ii) All the f_n are linear splines and $f_n(t) = 0$ in all the non-smoothness points of f_k, $k < n$, as well as in the points $0, 1/3, 2/3$ and 1.
(iii) Every $f_n|_{(2/3,1)}$ is linearly independent of $\{f_k|_{(2/3,1)}\}_{k=0}^{n-1}$.
(iv) The measure representing f_n^* is supported in $(2/3,1)$.
(v) $f_n^*(f_m) = 0$ when $n \neq m$, and $f_n^*(f_n) = 1$.

Let us explain the construction. Since $f_0|_{(2/3,1)} \neq 0$, we can select f_0^* supported in $(2/3,1)$ with $f_0^*(f_0) = 1$. Now we are going to select f_1. The conditions (ii) and (v) on f_1 mean that the linear spline f_1 must satisfy a finite number of linear equations:

$$f_1(0) = f_1(1/3) = f_1(2/3) = 0,\ f_0^*(f_1) = 0,$$

so the set of splines supported on a fixed non-void interval satisfying these conditions is a finite-codimensional subspace. Select a norm-one spline $g_1 \in C[0,1]$ supported on Δ_1 satisfying the equations above. Find a spline $h_1 \in C[0,1]$ of norm less than ε_1, supported on $(2/3,1) \setminus \Delta_1$, linearly independent of $f_0|_{(2/3,1)\setminus\Delta_1}$ and also satisfying the linear equations for f_1. Then $f_1 := g_1 + h_1$ will serve its purpose. By linear independence of $f_0|_{(2/3,1)\setminus\Delta_1}$ and $f_1|_{(2/3,1)\setminus\Delta_1} = h_1$, we may find a measure supported on $(2/3,1)$ (and even more: on $(2/3,1)\setminus\Delta_1$) which annihilates f_0 and takes the value 1 on f_1. Take this measure as f_1^*. Then, in the same way we construct f_2, then f_2^*, etc.

Now, we take $X := \overline{\mathrm{lin}}\{f_n\}_{n\in\mathbb{N}\cup\{0\}}$. The property (i) ensures that X is C-rich in $C[0,1]$. Thanks to the property (ii), $\{f_n\}_{n\in\mathbb{N}\cup\{0\}}$ forms a monotone basic sequence (i.e. a basis of X), and property (v) gives us that the coordinate functionals can be written as restrictions of f_n^* to X.

Since X is C-rich, there is a function $g\in S_X$ which attains its norm only on $(0,1/3)$. Fix $x_0^*\in \mathrm{ext}(B_{X^*})$ with $x_0^*(g)=1$. Let $\mu\in S_{C[0,1]^*}$ be a measure representing x_0^*. Then, μ is automatically supported on $(0,1/3)$, so $x_0^*(f_0)=0$. What remains to prove is that $f_0\in\mathrm{ext}(B_X)$. To this end, we consider $h\in X$ such that $\|f_0\pm h\|=1$. Since $f_0(t)=1$ for $t\in(2/3,1)$, we have that $h=0$ on $(2/3,1)$. But then, $h=\sum_{n=0}^{\infty} f_n^*(h)f_n\equiv 0$. □

In the complex case, an easier example can be constructed.

Example 5.2. *There is a C-rich subspace X of the complex space $C(\mathbb{T})$ (in particular, X is lush and so $n(X)=1$), and extreme points $x_0^*\in B_{X^*}$ and $\phi_0\in B_X$ such that $x_0^*(\phi_0)=0$.*

Proof. Let $A(\mathbb{D})$ be the disk algebra, considered as a closed subspace of $C(\mathbb{T})$. Then, $A(\mathbb{D})$ is C-rich in $C(\mathbb{T})$. Indeed, let $\varphi(z)=\exp(z)/\mathrm{e}$ for every $z\in\mathbb{D}$. Then, $\|\varphi\|=|\varphi(1)|=1$ and there is no other point on $\mathbb{T}$ but $z=1$ where φ attains its norm. Then, the family

$$\mathcal{A}=\{\varphi^n(z_0\,\cdot)\ :\ n\in\mathbb{N},\ z_0\in\mathbb{T}\}$$

belongs to $A(\mathbb{D})$, and for every $\varepsilon>0$ and every open subset U of $\mathbb{T}$, we may find an element of $\mathcal{A}$ which is at ε-distance from a function whose support is inside U.

Let $X=\mathrm{lin}\{A(\mathbb{D}),\ \phi_0\}$, where ϕ_0 is any function in $C(\mathbb{T})$ for which there are non-empty open sets U_1, U_2 and U_3 of $\mathbb{T}$ such that $\phi_0\equiv 1$ on U_1, $\phi_0\equiv -1$ on U_2 and $\phi_0\equiv 0$ on U_3. Then, X is C-rich because it contains $A(\mathbb{D})$. Next, ϕ_0 is extreme on B_X. Indeed, if $g=\alpha\phi_0+f\in X$ is such that $\|\phi_0\pm g\|\leqslant 1$, then $g\equiv 0$ on $U_1\cup U_2$, so $f\equiv\alpha$ on U_1 and $f\equiv -\alpha$ on U_2. It follows that $\alpha=0$ and so $g=0$. Also, for every $z\in\mathbb{T}$, the functional δ_z is extreme in B_{X^*}. Namely, for every $z\in\mathbb{T}$ there is a function $\varphi\in A(\mathbb{D})$ which attains its norm only at the point z. Then, there is an extreme point x^* of B_{X^*} such that $|x^*(\varphi)|=1$. Then, the norm-one measure which represents x^* must be supported on z (otherwise the integral would be strictly smaller than 1), so x^* is of the form $\theta\delta_z$. Finally, taking $z\in U_2$ and calling $x_0^*=\delta_z$, we have that $x_0^*\in\mathrm{ext}(B_{X^*})$ and $x_0^*(\phi_0)=0$. □

Let us comment that the extreme points f_0 and ϕ_0 of the examples above are not rotund. In fact, we do not know if a rotund point may exist in a lush space with dimension greater than one. We recall that a point x in the unit sphere of a Banach space X is said to be *rotund* if it is not an element of any nontrivial closed segment in the unit sphere or, equivalently, if $\|x+y\|=2$ for some $y\in B_X$ implies $y=x$.

To finish the section, we show that in the previous examples the extreme points of the unit ball cannot be w^*-extreme. We will use this to show that there are no lush spaces which are WMLUR.

Proposition 5.3. *Let X be a lush space. Then, for every w^*-extreme point x_0 of B_X and every $x^*\in\mathrm{ext}(B_{X^*})$, one has $|x^*(x_0)|=1$. In particular, this happens for WMLUR points of B_X.*

Proof. We fix a w^*-extreme point x_0 of B_X. By Lemma 4.2, there is a subset K_{x_0} of $\mathrm{ext}(B_{X^*})$ norming for X such that $x_0\in\overline{\mathrm{aco}(S(k^*,\varepsilon))}$ for every $\varepsilon>0$ and every $k^*\in K_{x_0}$. Then, since x_0 is an extreme point of the bidual ball, the same argument as at the end of the proof of Theorem 4.3 shows that for all $k^*\in K_{x_0}$

$$|x_0(k^*)|=1.$$

Since K_{x_0} is norming for X, we have that B_{X^*} is the weak*-closure of $\operatorname{aco}(\tilde{K})$, and the reversed Krein-Milman theorem gives us that the set $\operatorname{ext}(B_{X^*})$ is contained in the w^*-closure of $\mathbb{T}K_{x_0}$. The result follows since $x_0 \in X$. □

As a consequence, we have the following prohibitive result. In the real case, it is a particular case of Corollary 4.6, since WMLUR spaces are strictly convex.

Corollary 5.4. *Let X be a lush space. Then, X is not WMLUR (in particular, it is not MLUR), unless it is one-dimensional.*

Proof. Since X is a WMLUR, Proposition 5.3 gives us that $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$ and every $x \in S_X$. But this clearly implies that X is one-dimensional. □

Let us mention that another consequence of Proposition 5.3 is that every w^*-extreme point of a lush space is actually MLUR, as the following remark shows.

Remark 5.5. *Let X be a Banach space and let x be a point in B_X so that $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$. Then, x is an MLUR point of B_X. Indeed, fixed $y \in X$, we take $x^* \in \operatorname{ext}(B_{X^*})$ so that $x^*(y) = \|y\|$ and we estimate as follows*

$$\max_{\pm}\|x \pm y\| \geqslant \max_{\pm}\big|x^*(x) \pm \|y\|\big| \geqslant \big(|x^*(x)|^2 + \|y\|^2\big)^{1/2} = (1 + \|y\|^2)^{1/2}.$$

Finally, we present an example showing that the results above are not valid for Banach spaces having the alternative Daugavet property. We do not know whether they are true for spaces with numerical index 1.

Example 5.6. *The real or complex space $X = C\big([0,1], \ell_2^2\big)$ has the alternative Daugavet property (and even the Daugavet property). However, there exist $x_0^* \in \operatorname{ext}(B_{X^*})$ and a MLUR point f_0 of B_X such that $|x_0^*(f_0)| < 1$.*

Proof. First, $C([0,1], \ell_2^2)$ has the alternative Daugavet property by [36, Theorem 3.4], for instance. Now, we fix any $x_0 \in S_{\ell_2^2}$ and consider $f_0 \in S_X$ given by $f_0(t) = x_0$ for every $t \in [0,1]$. To prove that f_0 is an MLUR point in B_X, we take $g \in X$ and we observe that

$$\max_{\pm}\|f_0 \pm g\| = \sup_{t\in[0,1]} \max_{\pm}\|x_0 \pm g(t)\| \geqslant \sup_{t\in[0,1]} \big(1 + \|g(t)\|^2\big)^{1/2} = (1 + \|g\|^2)^{1/2}.$$

We notice that the above inequality becomes an equality when one considers $g \in X$ given by $g(t) = x_0^{\perp}$ for every $t \in [0,1]$, where $x_0^{\perp} \in S_{\ell_2^2}$ is orthogonal to x_0. Finally, it suffices to take $x_0^* \in \operatorname{ext}(B_{X^*})$ so that $x_0^*(g) = 1$ to get the desired condition. Indeed,

$$\sqrt{2} = \max_{\omega\in\mathbb{T}}\|f_0 + \omega\, g\| \geqslant \max_{\omega\in\mathbb{T}}|x_0^*(f_0) + \omega| = 1 + |x_0^*(f_0)|,$$

so $|x_0^*(f_0)| \leqslant \sqrt{2} - 1 < 1$. □

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(Kadets) DEPARTMENT OF MECHANICS AND MATHEMATICS, KHARKOV NATIONAL UNIVERSITY, PL. SVOBODY 4, 61077 KHARKOV, UKRAINE

E-mail address: vova1kadets@yahoo.com

(Martín, Merí, Payá) DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA, 18071 GRANADA, SPAIN

E-mail address: mmartins@ugr.es, jmeri@ugr.es, rpaya@ugr.es