POLYNOMIAL NUMERICAL INDICES OF $C(K)$ AND $L_1(\mu)$

DOMINGO GARCÍA, BOGDAN C. GRECU, MANUEL MAESTRE, MIGUEL MARTÍN, AND JAVIER MERÍ

ABSTRACT. We estimate the polynomial numerical indices of the spaces $C(K)$ and $L_1(\mu)$.

1. Introduction

For a given real or complex Banach space X and a positive integer k, the *k-order polynomial numerical index* of X was introduced by Choi et al. in 2006 [2] as follows

$$\begin{aligned} n^{(k)}(X) &= \inf\left\{v(P) \,:\, P \in \mathcal{P}\left({}^kX; X\right),\ \|P\| = 1\right\} \\ &= \max\left\{k \geqslant 0 \,:\, k\|P\| \leqslant v(P) \ \text{ for all } \ P \in \mathcal{P}\left({}^kX; X\right)\right\}. \end{aligned}$$

$\mathcal{P}\left({}^kX; X\right)$ denotes the space of k-homogeneous polynomials and $v(\cdot)$ is the *numerical radius*, which is defined as

$$v(P) = \sup\left\{|x^*(P(x))| \,:\, x \in X,\, x^* \in X^*,\, \|x\| = \|x^*\| = 1\right\}$$

for $P \in \mathcal{P}\left({}^kX; X\right)$ (X^* represents the dual space to X). When $k = 1$, we are actually dealing with the classical numerical radius of operators due to Lumer and Bauer in the 1960's and with the index of a Banach space introduced by Lumer in 1970. The extension of the numerical radius to polynomials and other settings was initiated by Harris in the 1970's. We refer the reader to the survey paper [7] for a detailed account and background on numerical radii and numerical indices. More recent results on polynomial numerical indices can be found in [3, 5, 8, 9, 11, 12, 13].

Some spaces for which the polynomial numerical indices have been estimated are the following. In the complex case, $n^{(k)}(C_0(L)) = 1$ for every $k \in \mathbb{N}$ and every locally compact space L, and $n^{(2)}(\ell_1) \leqslant \frac{1}{2}$. In the real case, $n^{(k)}(\mathbb{R}) = 1$ and

$$n^{(2)}(c_0) = n^{(2)}(\ell_1) = n^{(2)}(\ell_\infty) = 1/2.$$

The main results of this paper are the following. In the real case, $n^{(2)}(C(K)) = 1/2$ for every compact Hausdorff space K with at least two points and $n^{(2)}(L_1(\mu)) = 1/2$ for every positive measure μ. In the complex case, $n^{(2)}(X) \geqslant 1/3$ for every lush space and $1/3 \leqslant n^{(2)}(L_1(\mu)) \leqslant 1/2$ for every positive measure μ.

Given a compact Hausdorff space K, we denote by $C(K)$ the Banach space of all continuous functions from K into $\mathbb{R}$ or $\mathbb{C}$. For a locally compact Hausdorff space L, we denote by $C_0(L)$ the Banach space of all continuous functions from L into $\mathbb{R}$ or $\mathbb{C}$ vanishing at infinity. Also, given a measure space (Ω, Σ, μ), we denote by $L_1(\mu)$ the Banach space of all (equivalence classes of) integrable functions on (Ω, Σ, μ). As usual, c_0, ℓ_1 and ℓ_∞ will denote the classical Banach spaces of all null, absolutely summable and bounded sequences, respectively.

Date: October 31, 2011.

2010 *Mathematics Subject Classification.* Primary 46B04; Secondary 46B20, 46G25, 47A12.

Key words and phrases. Polynomial, Banach space, numerical range, polynomial numerical index.

The first and third authors were supported by MICINN grant MTM2011-22417. The third author was also supported by Prometeo 2008/101. This article was started while the second author was a postdoctoral fellow at Departamento de Análisis Matemático, Universidad de Valencia. He was supported by a Marie Curie Intra European Fellowship (MEIF-CT-2005-006958). The fourth and fifth authors were supported by Spanish MICINN and FEDER grant MTM2009-07498 and Junta de Andalucía and FEDER grants G09-FQM-185 and P09-FQM-4911.

2. The results

Our first aim is to show that, in the real case, $n^{(2)}(C(K)) = \frac{1}{2}$ for every compact Hausdorff space with at least two points. We may also get some estimations on the polinomial numerical indices of higher degree of real $C(K)$ spaces improving the results obtained in [9, Corollary 2.5].

Theorem 1. *Let K be a compact Hausdorff space with at least two points. Then*

$$n^{(2)}(C(K)) = \frac{1}{2} \qquad \text{and} \qquad n^{(2k)}(C(K)) \leqslant \frac{1}{2^k} \quad (k \geqslant 2)$$

hold in the real case.

Proof. Let us start with the case of degree 2. By [9, Corollary 2.4], $n^{(2)}(C(K)) \geqslant \frac{1}{2}$. To prove the other inequality, we fix a point $t_0 \in K$, consider the polynomial of degree two on $C(K)$ given by

$$P(f) = f(t_0)^2\, \mathbf{1} - \frac{1}{2} f^2 \qquad (f \in C(K)),$$

where $\mathbf{1}$ stands for the unit function, and we observe that $\|P\| \geqslant 1$ and $v(P) = \frac{1}{2}$. Indeed, we consider a point $t_1 \in K \setminus \{t_0\}$ and construct a norm-one function $f \in C(K)$ satisfying $f(t_0) = 1$ and $f(t_1) = 0$. Then,

$$\|P(f)\| \geqslant |P(f)(t_1)| = 1.$$

On the other hand, since all elements in $C(K)$ attain their norms, we have (see [14, Theorem 2.5]) that

$$v(P) = \sup\{|P(f)(t)| \ : \ f \in C(K),\ t \in K,\ \|f\| = f(t) = 1\}.$$

Therefore, in our case,

$$v(P) = \sup\left\{ \left| f(t_0)^2 - \frac{1}{2} \right| \ : \ f \in C(K),\ \|f\| = 1 \right\} = \frac{1}{2},$$

which finishes the proof.

For higher degrees, we fix $k \in \mathbb{N}$ with $k \geqslant 2$, consider the polinomial $P_k \in \mathcal{P}(^{2k}C(K), C(K))$ given by $P_k(f) = P(f)^k$ for every $f \in C(K)$ (where P is the above defined polynomial), and we observe that $\|P_k\| \geqslant 1$ and $v(P_k) = \dfrac{1}{2^k}$. □

With the same proof, considering bump functions instead of the unit, the result above adapts to $C_0(L)$ for every locally compact Hausdorff space L with at least two points.

Corollary 2. *Let L be a locally compact Hausdorff space L with at least two points. Then*

$$n^{(2)}(C_0(L)) = \frac{1}{2} \qquad \text{and} \qquad n^{(2k)}(C_0(L)) \leqslant \frac{1}{2^k} \quad (k \geqslant 2)$$

hold in the real case.

Remark 3. *It it easy to extend the proof of Theorem 1 to any closed subalgebra with dimension greater than one of a real space $C(K)$* . But, actually, all closed subalgebras of a real $C(K)$ space are of the form

$$\{f \in C(K) \ : \ f(t_i) = \lambda_i f(s_i) \ \text{ for all } \ i \in I\}$$

for a suitable index set I and suitable families $\{t_i\}, \{s_i\} \subset K$ and $\{\lambda_i\} \subset \{0, 1\}$ (see [10, p. 68]).

Our next result deals with lush spaces. For a Banach space X, by B_X and S_X we will denote the open unit ball and the unit sphere of X, respectively. A Banach space X is said to be *lush* [1] if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice $S = \{x \in B_X \ : \ \operatorname{Re} x^*(x) > 1 - \varepsilon\}$ with $x^* \in S_{X^*}$ such that $x \in S$ and the distance of y to the absolutely convex hull of S is less than ε. Lush spaces have numerical index 1 [1, Proposition 2.2], but it has been very recently shown that the converse result is not true [6]. Examples of lush spaces are $L_1(\mu)$ spaces and their isometric preduals, in particular, $C(K)$ spaces. In [9], inequalities for the polynomial numerical indices of real lush spaces were given. In particular, it is proved that $n^{(2)}(X) \geqslant 1/2$ for every real lush space and that the equality holds for c_0, ℓ_1 and ℓ_∞, for instance. Actually, our Theorem 1 gives that such an equality also holds for all $C(K)$ spaces. Our next goal is to give a similar result to the one of [9] for complex lush spaces. We do not know whether this result is sharp.

Theorem 4. *Let X be a complex lush space. Then*

$$n^{(2)}(X) \geqslant \frac{1}{3}\,.$$

Proof. For $P \in \mathcal{P}\left({}^2X; X\right)$ with $\|P\| = 1$ and $0 < \varepsilon < 1$ fixed, take $x_0 \in S_X$ such that $\|P(x_0)\| > 1 - \varepsilon$, and apply the definition of lushness to x_0 and $\frac{P(x_0)}{\|P(x_0)\|}$ to find $x^* \in S_{X^*}$ with

$$\frac{P(x_0)}{\|P(x_0)\|} \in S := \{x \in B_X \,:\, \operatorname{Re} x^*(x) > 1 - \frac{\varepsilon^2}{4}\}\,,$$

$\lambda_1, \ldots, \lambda_n \in [0,1]$, $\mu_1, \ldots, \mu_n$, complex numbers of modulus one, and $x_1, \ldots, x_n \in S$ satisfying

$$\Big\| x_0 - \sum_{j=1}^{n} \lambda_j \mu_j x_j \Big\| < \frac{\varepsilon^2}{4}\,.$$

With this it is readily checked that

$$\Big\| P(x_0) - P\Big(\sum_{j=1}^{n} \lambda_j \mu_j x_j\Big) \Big\| \leqslant 2\|\check{P}\| \frac{\varepsilon^2}{4}\,, \tag{1}$$

where $\check{P}$ is the associated symmetric bilinear map to P.

Our goal is to estimate $\left| x^*\left(P\left(\sum_{j=1}^n \lambda_j\mu_j x_j\right)\right)\right|$ from above and below. We can write

$$\left| x^*\left(P\left(\textstyle\sum_{j=1}^n \lambda_j\mu_j x_j\right)\right)\right| \leqslant \sum_{j=1}^{n} \lambda_j^2 \big|x^*(P(x_j))\big| + 2 \sum_{1 \leqslant j < k \leqslant n} \lambda_j \lambda_k \big|x^*(\check{P}(x_j, x_k))\big|.$$

Moreover, by using [9, Lemma 2.3] one obtains

$$\big|x^*(P(y))\big| \leqslant v(P) + \varepsilon + 2\,\|\check{P}\|\,\varepsilon$$

for every $y \in S$. This, together with the fact that $x_j, \frac{x_j + x_k}{2} \in S$, tells us

$$\big|x^*(\check{P}(x_j, x_k))\big| \leqslant 2\left|x^*\left(P\left(\tfrac{x_j+x_k}{2}\right)\right)\right| + \frac{1}{2}\big|x^*(P(x_j))\big| + \frac{1}{2}\big|x^*(P(x_k))\big| \leqslant 3\big(v(P) + \varepsilon + 2\,\|\check{P}\|\,\varepsilon\big).$$

Hence, we can continue the above estimation as follows:

$$\begin{aligned}
\left| x^*\left(P\left(\textstyle\sum_{j=1}^n \lambda_j\mu_j x_j\right)\right)\right| &\leqslant \left(\sum_{j=1}^{n} \lambda_j^2 + 6 \sum_{1 \leqslant j < k \leqslant n} \lambda_j \lambda_k\right)\big(v(P) + \varepsilon + 2\,\|\check{P}\|\,\varepsilon\big) \\
&\leqslant \sup_{\substack{\lambda_j \in [0,1] \\ \lambda_1 + \cdots + \lambda_n = 1}} \left(\sum_{j=1}^{n} \lambda_j^2 + 6 \sum_{1 \leqslant j < k \leqslant n} \lambda_j \lambda_k\right)\big(v(P) + \varepsilon + 2\,\|\check{P}\|\,\varepsilon\big) \\
&= \left(3 - \frac{2}{n}\right)\big(v(P) + \varepsilon + 2\,\|\check{P}\|\,\varepsilon\big) \\
&\leqslant 3\big(v(P) + \varepsilon + 2\,\|\check{P}\|\,\varepsilon\big).
\end{aligned}$$

On the other hand, using (1) we have that

$$\begin{aligned}
\left| x^*\left(P\left(\textstyle\sum_{j=1}^n \lambda_j\mu_j x_j\right)\right)\right| &\geqslant |x^*(P(x_0))| - \left|x^*(P(x_0)) - x^*\left(P\left(\textstyle\sum_{j=1}^n \lambda_j\mu_j x_j\right)\right)\right| \\
&= \|P(x_0)\| \left| x^*\left(\frac{P(x_0)}{\|P(x_0)\|}\right)\right| - \left|x^*(P(x_0)) - x^*\left(P\left(\textstyle\sum_{j=1}^n \lambda_j\mu_j x_j\right)\right)\right| \\
&\geqslant (1 - \varepsilon)\left(1 - \frac{\varepsilon^2}{4}\right) - 2\|\check{P}\|\frac{\varepsilon^2}{4}\,.
\end{aligned}$$

Therefore,

$$(1 - \varepsilon)\left(1 - \frac{\varepsilon^2}{4}\right) - 2\|\check{P}\|\frac{\varepsilon^2}{4} \leqslant 3\big(v(P) + \varepsilon + 2\,\|\check{P}\|\,\varepsilon\big)$$

which, letting $\varepsilon \to 0$, gives $\frac{1}{3} \leqslant v(P)$ finishing the proof. ∎

For $L_1(\mu)$ spaces, we get the following result.

Theorem 5. *Let (Ω,Σ,μ) be a measure space so that $\dim\big(L_1(\mu)\big)\geqslant 2$. Then the following hold*

$$n^{(2)}\big(L_1(\mu)\big)=\frac{1}{2}\quad \text{in the real case}\qquad \text{and}\qquad \frac{1}{3}\leqslant n^{(2)}\big(L_1(\mu)\big)\leqslant\frac{1}{2}\quad\text{in the complex case.}$$

Proof. We start by showing that $n^{(2)}\big(L_1(\mu)\big)\leqslant\frac{1}{2}$. To do so, we distinguish two cases depending on wether μ is purely atomic or not. If μ is purely atomic, since $\dim\big(L_1(\mu)\big)\geqslant 2$, we can write $L_1(\mu)=\ell_1(\Gamma)$ with the cardinal of Γ greater or equal than 2 and then $n^{(2)}\big(L_1(\mu)\big)\leqslant\frac{1}{2}$ (see [4, p. 141]). If otherwise μ has non-atomic part then it is possible to find disjoint sets $A,B\in\Sigma$ satisfying $0<\mu(A)=\mu(B)<\infty$. Thus, we can consider the polynomial $P\in\mathcal{P}\big({}^2L_1(\mu),L_1(\mu)\big)$ given by

$$P(f)=\left(\frac{1}{2}f\int_A f\,d\mu+2f\int_B f\,d\mu\right)\chi_A+\left(-f\int_A f\,d\mu-\frac{1}{2}f\int_B f\,d\mu\right)\chi_B\qquad \big(f\in L_1(\mu)\big)$$

which satisfies $\|P\|\geqslant 1$ and $v(P)\leqslant\frac{1}{2}$. Indeed, for $f=\frac{1}{2\mu(A)}\chi_{A\cup B}\in S_{L_(\mu)}$ it is immediate to check that

$$P(f)=\frac{5}{8\mu(A)}\chi_A-\frac{3}{8\mu(A)}\chi_B$$

and so, $\|P\|\geqslant\|P(f)\|=1$. To estimate $v(P)$, given $f\in S_{L_1(\mu)}$ and $\Phi\in S_{L_1(\mu)^*}$ with $\Phi(f)=1$, we write ϕ for the unique element in $S_{L_\infty(\mu)^*}$ which represents Φ (i.e. $\Phi(h)=\int\phi h\,d\mu$ for every $h\in L_1(\mu)$) and observe that

$$\int_A f\phi\,d\mu=\int_A|f|\,d\mu\qquad\text{and}\qquad\int_B f\phi\,d\mu=\int_B|f|\,d\mu.$$

Hence, we can write

$$\begin{aligned}|\Phi(Pf)|&=\left|\left(\frac{1}{2}\int_A f\,d\mu+2\int_B f\,d\mu\right)\int_A f\phi\,d\mu-\left(\int_A f\,d\mu+\frac{1}{2}\int_B f\,d\mu\right)\int_B f\phi\,d\mu\right|\\&=\left|\left(\frac{1}{2}\int_A|f|\,d\mu-\int_B|f|\,d\mu\right)\int_A f\,d\mu+\left(2\int_A|f|\,d\mu-\frac{1}{2}\int_B|f|\,d\mu\right)\int_B f\,d\mu\right|\\&\leqslant\left|\frac{1}{2}\int_A|f|\,d\mu-\int_B|f|\,d\mu\right|\int_A|f|\,d\mu+\left|2\int_A|f|\,d\mu-\frac{1}{2}\int_B|f|\,d\mu\right|\int_B|f|\,d\mu.\end{aligned}$$

Now, since $\displaystyle\int_A|f|\,d\mu+\int_B|f|\,d\mu\leqslant 1$, we have

$$|\Phi(Pf)|\leqslant\max_{\substack{x,y\geqslant 0\\ x+y\leqslant 1}}\left|\frac{1}{2}x-y\right|x+\left|2x-\frac{1}{2}y\right|y=\frac{1}{2}$$

which gives $v(P)\leqslant\frac{1}{2}$ and, therefore, $n^{(2)}\big(L_1(\mu)\big)\leqslant\frac{1}{2}$. Finally, the remaining inequalities follow from [9, Theorem 2.1], Theorem 4, and the fact that $L_1(\mu)$ is a lush space. ☐

References

[1] K. Boyko, V. Kadets, M. Martín, and D. Werner, Numerical index of Banach spaces and duality, *Math. Proc. Cambridge Phil. Soc.* **142** (2007), 93–102.

[2] Y. S. Choi, D. García, S. G. Kim and M. Maestre, The polynomial numerical index of a Banach space, *Proc. Edinb. Math. Soc.* **49** (2006), 39–52.

[3] Y. S. Choi, D. García, M. Maestre, and M. Martín, Polynomial numerical index for some complex vector-valued function spaces, *Quart. J. Math.* **59** (2008), 455–474.

[4] Y. S. Choi and S. G. Kim, Norm or numerical radius attaining multilinear mappings and polynomials, *J. London Math. Soc.* **54** (1996), no. 2, 135–147.

[5] D. García, B. Grecu, M. Maestre, M. Martín and J. Merí, Two-dimensional Banach spaces with polynomial numerical index zero, *Linear Algebra Appl.* **430** (2009), 2488–2500.

[6] V. Kadets, M. Martín, J. Merí, and V. Shepelska, Lushness, numerical index 1 and duality, *J. Math. Anal. Appl.* **357** (2009), 15–24.

[7] V. Kadets. M. Martín, and R. Payá, Recent progress and open questions on the numerical index of Banach spaces, *Rev. R. Acad. Cien. Serie A. Mat.* **100** (2006), 155–182.

[8] J. Kim and H. J. Lee, Strong peak points and strongly norm attaining points with applications to denseness and polynomial numerical indices, *J. Funct. Anal.* **257** (2009), 931–947.

[9] S. G. Kim, M. Martín, and J. Merí, On the polynomial numerical index of the real spaces c_0, ℓ_1 and ℓ_∞, *J. Math. Anal. Appl.* **337** (2008), 98–106.

[10] H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, Berlin 1972.

[11] H. J. Lee, Banach spaces with polynomial numerical index 1, *Bull. London Math. Soc.* **40** (2008), 193–198.

[12] H. J. Lee and M. Martín, Polynomial numerical indices of Banach spaces with 1-unconditional basis, *preprint.*

[13] H. J. Lee, M. Martín, and J. Merí, Polynomial numerical indices of Banach spaces with absolute norm, *Linear Algebra Appl.* **435** (2011), 400–408.

[14] A. Rodríguez-Palacios, Numerical ranges of uniformly continuous functions on the unit sphere of a Banach space, *J. Math. Anal. Appl.* **2**97 (2004), no. 2, 472–476.

(Grecu) School of Mathematics and Physics, Queen's University Belfast, BT7 1NN, United Kingdom

E-mail address: b.grecu@qub.ac.uk

(García & Maestre) Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

E-mail address: domingo.garcia@uv.es, manuel.maestre@uv.es

(Martín & Merí) Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

E-mail address: mmartins@ugr.es, jmeri@ugr.es