PROPERTIES OF LUSH SPACES AND APPLICATIONS TO BANACH SPACES WITH NUMERICAL INDEX 1

KOSTYANTYN BOYKO, VLADIMIR KADETS, MIGUEL MARTÍN, AND JAVIER MERÍ

Abstract. The concept of lushness was introduced recently as a Banach space property, which ensures that the space has numerical index 1. We prove that for Asplund spaces lushness is actually equivalent to numerical index 1. We prove that every separable Banach space containing an isomorphic copy of c_0 can be renormed equivalently to be lush, and thus to have numerical index 1. The rest of the paper is devoted to the study of lushness just as a property of Banach spaces. We prove that lushness is separably determined, is stable under ultraproducts, and we characterize those spaces of the form $X = (\mathbb{R}^n, \|\cdot\|)$ with absolute norm such that X-sum preserves lushness of summands, showing that this property is equivalent to lushness of X.

1. Introduction

Let us fix first some notations. All over the paper X stands for a Banach space, B_X and S_X are, respectively, its closed unit ball and its unit sphere, X^* is the dual space to X, and $L(X)$ is the Banach algebra of bounded linear operators on X. All linear spaces are over the field $\mathbb{K}$, which can be either the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers. For a functional $x^* \in S_{X^*}$ and a positive number α, the set of the form

$$S(B_X, x^*, \alpha) = \{x \in B_X \ : \ \operatorname{Re} x^*(x) > 1 - \alpha\}$$

is called a *slice* of the unit ball. If A is a subset of X, we write $\operatorname{conv}(A)$ for the convex hull of A and $\operatorname{aconv}(A)$ for the absolutely convex hull of A. Finally, we denote by $\operatorname{ext}(A)$ the set of extreme points of the convex subset $A \subseteq X$. Now the basic definition of our paper:

Definition 1.1. A Banach space X is said to be *lush* if for every $x, y \in S_X$ and for every $\varepsilon > 0$ there is a slice $S = S(B_X, x^*, \varepsilon) \subset B_X$, $x^* \in S_{X^*}$, such that $x \in S$ and $\operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon$.

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Among the examples of lush spaces are the real or complex spaces $C(K)$ and $L_1(\mu)$ (see [5] or [9, §3] for a detailed account).

The concept of lushness was introduced recently in [5] as a geometrical property of a Banach space which ensures that the space has numerical index 1. The *numerical index* of a Banach space X was introduced by G. Lumer in 1968 (see [6]) and it is the best constant of equivalence between the numerical radius and the usual norm of operators on $L(X)$. Concretely,

$$\begin{aligned} n(X) &= \inf\{v(T) \ : \ T \in L(X),\ \|T\| = 1\} \\ &= \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \ \forall T \in L(X)\}. \end{aligned}$$

Here, for $T \in L(X)$, $v(T)$ is its numerical radius:

$$v(T) = \sup\bigl\{|x^*(T(x))| \ : \ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\bigr\}.$$

Thus, a Banach space has numerical index 1 iff $v(T) = \|T\|$ for every $T \in L(X)$. It is clear from the definition that $0 \leqslant n(X) \leqslant 1$, being these inequalities best possible in the real case. In the complex case, it is a deep result that $n(X)$ is always bigger or equal than $1/\mathrm{e}$. Classical references here are the monographs from the 1970's [2, 3]. The state-of-the-art on numerical indices may be found in the recent survey [9] and references therein.

The concept of lushness is proven to be a useful tool in the theory of numerical index of Banach spaces since in [5] it helped to construct an example showing that numerical index is not inherited in general by the dual space, a latent question in the theory from the beginning of the subject. Also, in [10] the lushness was applied for estimating the related concept of polynomial numerical index in some real spaces like c_0 or ℓ_1.

One major difficulty when dealing with Banach spaces with numerical index 1 is that the definition involves operators, and one has to deal with all operators, since the analogous property with only compact operators (called *the alternative Daugavet property*) is known to be strictly weaker (see [15]). On the other hand, there are in the literature many isometric properties which are sufficient conditions for a Banach space to have numerical index 1, being lushness the weakest of all of them (see [9, §3] for a detailed account). In [14], it is shown that many of these properties are actually equivalent to numerical index 1 for Banach spaces with the Radon-Nikodým property. But this is not true for general Banach spaces (see [5, Example 3.4]). In this paper we prove that Asplund spaces with numerical index 1 are lush (section 2).

Section 3 is devoted to renorming. We prove that every separable Banach space containing an isomorphic copy of c_0 can be equivalently renormed to be lush, and thus to have numerical index 1. Up to our knowledge, this is the first non-trivial sufficient condition for renorming with numerical index 1. Let us say that 1 is the only interesting value of the numerical index from the isomorphic point of view: in [7] it was proved that "many" Banach spaces (for instance, reflexive or separable spaces) can be equivalently renormed to have any possible value of the numerical index smaller than 1 (i.e. any number in $[0,1[$ in the real case and any number in $[1/\mathrm{e},1[$ in the complex case). On the other hand, some necessary conditions for a Banach space to be renormed with numerical index 1 were given in [12].

The rest of the paper is devoted to the study of lushness independently of its applications to numerical index. We show in section 4 a reformulation of lushness which allows us to prove that it is separably determined and that it is inherited by ultraproducts. Finally, section 5 is devoted to characterize those spaces of the form $X=(\mathbb{R}^n,\|\cdot\|)$ with absolute norm such that X-sum preserves lushness of summands (we show that this property is actually equivalent to lushness of X).

We have to stress out that the study of lush spaces is still in its "embryonal" faze, and that the number of open questions is much bigger than that of results obtained.

2. Lushness and numerical index 1

In this section we give sufficient conditions for lushness which will be useful in the rest of the paper. Some notation is needed. Given a Banach space X, a subset $A\subset S_{X^*}$ is said to be *norming* for X if

$$\|x\| = \sup\{|a^*(x)| \,:\, a^*\in A\}$$

for every $x\in X$. Given a completely regular Hausdorff topological space Ω, we write $C_b(\Omega)$ to denote the Banach space of all $\mathbb{K}$-valued bounded continuous functions on Ω, endowed with the supremum norm. Finally, we recall the fact that X is Asplund if and only if X^* has the Radon-Nikodým property.

Theorem 2.1. *Let X be a Banach space. We consider the following assertions.*

- (a) *There is a completely regular Hausdorff topological space Ω and an isometric embedding $J: X\longrightarrow C_b(\Omega)$ such that $|x^{**}(J^*(\delta_s))|=1$ for every $s\in\Omega$ and $x^{**}\in \operatorname{ext}(B_{X^{**}})$,*
- (b) *There is a norming set $A\subset B_{X^*}$ for X such that $|x^{**}(a^*)|=1$ for every $a^*\in A$ and every $x^{**}\in\operatorname{ext}(B_{X^{**}})$,*
- (c) *For each $x\in S_X$ and $\varepsilon>0$ there exists $x^*\in S_{X^*}$ such that*

$$x\in S=S(B_X,x^*,\varepsilon) \qquad \text{and} \qquad B_X=\overline{\operatorname{aconv}(S)},$$

- (d) *X is lush.*
- (e) *$n(X)=1$.*

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

In the case when X is an Asplund space, then all the assertions above are equivalent.

Proof. $(a) \Rightarrow (b)$. It suffices to show that $\{J^*(\delta_s) : s \in \Omega\}$ is norming. To do so, fixed $x \in X$ we observe that

$$\sup_{s\in\Omega} [J^*(\delta_s)](x) = \sup_{s\in\Omega} \delta_s(Jx) = \|Jx\| = \|x\|.$$

$(b) \Rightarrow (a)$. We consider the operator $J : X \longrightarrow C_b(A)$ defined by

$$[Jx](a^*) = a^*(x) \qquad (x \in X,\ a^* \in A),$$

which is isometric since the set $A \subset B_{X^*}$ is norming. Besides, it is obvious from the definition of J that $J^*(\delta_{a^*}) = a^*$ for every $a^* \in A$ and, therefore, we have that

$$|x^{**}(J^*(\delta_{a^*}))| = 1$$

for every $a^* \in A$ and every $x^{**} \in \operatorname{ext}(B_{X^{**}})$.

$(b) \Rightarrow (c)$. Let $\varepsilon > 0$ and $x \in S_X$ be fixed. We take $x^* \in A$ such that $\operatorname{Re} x^*(x) > 1 - \varepsilon$, we define $S = S(B_X, x^*, \varepsilon)$, and we prove that $B_X = \overline{\operatorname{aconv}(S)}$. To do so, we define $\widetilde{S} = S(B_{X^{**}}, x^*, \varepsilon)$ and we observe that the hypothesis gives us

$$B_{X^{**}} = \overline{\operatorname{aconv}\big(\widetilde{S}\big)}^{w^*}.$$

On the other hand, it is clear that $\widetilde{S} \subset \overline{S}^{w^*}$, which tells us that

$$\overline{\operatorname{aconv}(S)}^{w^*} = \overline{\operatorname{aconv}\big(\widetilde{S}\big)}^{w^*} = B_{X^{**}}.$$

Finally, we deduce that

$$B_X = B_{X^{**}} \cap X = \overline{\operatorname{aconv}(S)}^{w^*} \cap X = \overline{\operatorname{aconv}(S)}^{w} = \overline{\operatorname{aconv}(S)}.$$

$(c) \Rightarrow (d)$ is completely evident and $(d) \Rightarrow (e)$ was proved in [5, Proposition 2.2].

Finally, the case when X is Asplund follows from [5, Remark 3.5]. □

Let us point out that numerical index 1 and lushness are also equivalent for Banach spaces with the Radon-Nikodým property. This is an immediate consequence of [14, Theorem 1] and [5, Proposition 2.2].

Remark 2.2. Let X be a Banach space with the Radon-Nikodým property. Then, X is lush if and only if $n(X) = 1$.

We are giving now two classes of spaces where Theorem 2.1 applies. The first class consists of preduals of $L_1(\mu)$ spaces. Indeed, it is clear that $\left|\int \varphi f \, d\mu\right| = 1$ for every $f \in \operatorname{ext}(B_{L_1(\mu)})$ and every $\varphi \in \operatorname{ext}(B_{L_\infty(\mu)})$. Now, if $L_1(\mu)$ has a predual X, then the set $\operatorname{ext}(B_{L_1(\mu)})$ is norming for X and condition (b) of Theorem 2.1 applies.

Example 2.3. The preduals of any $L_1(\mu)$ space are lush.

The second class of spaces in which Theorem 2.1 applies is the one of nicely embedded spaces in $C_b(\Omega)$ spaces. A Banach space X is said to be *nicely embedded* in $C_b(\Omega)$ if there exists a linear isometry $J : X \longrightarrow C_b(\Omega)$ such that for all $s \in \Omega$ the following properties are satisfied:

(N1) $\|J^*\delta_s\| = 1$.
(N2) $\operatorname{span}(J^*\delta_s)$ is an L-summand in X^*.

This property was introduced in [18], where the corresponding examples can be found. It is immediate that nicely embedded spaces fulfill condition (a) in Theorem 2.1, so they are lush.

Example 2.4. Any Banach space which nicely embeds into a $C_b(\Omega)$ space is lush.

We finish the section with a result concerning duals of Radon-Nikodým spaces. It is shown in [5, Proposition 4.1] that $n(X^*) = 1$ if X is a Banach space with the Radon-Nikodým property and $n(X) = 1$. Actually, with the help of Theorem 2.1, it can be proved that X^* is lush.

Proposition 2.5. *Let X be a Banach space with the Radon-Nikodým property and $n(X) = 1$. Then, for each $x^* \in S_{X^*}$ and $\varepsilon > 0$ there exists $x^{**} \in S_{X^{**}}$ such that*

$$x^* \in S = S(B_{X^*}, x^{**}, \varepsilon) \qquad \textit{and} \qquad B_{X^*} = \overline{\operatorname{aconv}(S)}.$$

In particular, X^* is lush.

Proof. Following the proof of [5, Proposition 4.1], one has that

$$|x^{***}(a)| = 1 \qquad \big(x^{***} \in \operatorname{ext}(B_{X^{***}}),\ a \in A\big)$$

where A is the set of denting points of B_X viewed as a subset of $B_{X^{**}}$. Since X has the Radon-Nikodým property, A is a norming subset of $B_{X^{**}}$ for X^* and the result follows from Theorem 2.1. □

3. Lush renormings

Our goal in this section is to prove that a separable Banach space containing an isomorphic copy of c_0 can be equivalently renormed to be lush (in particular, to have numerical index 1). We need two lemmata.

Lemma 3.1. *Let X be a separable Banach space containing an isometric copy of c_0. Then there is a biorthogonal system $\{(g_n, g_n^*)\} \subset B_X \times (12 B_{X^*})$ such that*

$$\sup_{n\in\mathbb{N}} |g_n^*(x)| \geqslant \frac{1}{3}\|x\| \tag{1}$$

for all $x \in X$.

Proof. Since a c_0-subspace of a separable space is 2-complemented (Sobczyk's Theorem, see [1, Corollary 2.5.9] for instance), one can write down X as $c_0 \oplus Y$ in such a way, that for every $e \in c_0$, $y \in Y$

$$\frac{1}{6}\left(\|e\| + \|y\|\right) \leqslant \|e + y\| \leqslant \|e\| + \|y\|. \tag{2}$$

Denote by $\{e_n\}_{n\in\mathbb{N}}$ the canonical basis of c_0 and by $\{e_n^*\}_{n\in\mathbb{N}} \subset Y^\perp \subset X^*$ denote the corresponding coordinate functionals. By (2), $\|e_n^*\| \leqslant 6$ for every $n \in \mathbb{N}$. Now, we use the separability of Y to take a norming sequence with norming tails $\{y_n^*\}_{n\in\mathbb{N}} \subset S_{Y^*}$, that is

$$\sup_{n\geqslant m} |y_n^*(y)| = \|y\| \qquad \big(y \in Y,\ m \in \mathbb{N}\big).$$

We write $\widetilde{y}_n^* \in c_0^\perp \subset X^*$ for the natural extensions of y_n^* to the whole of X. Again, by (2), $\|\widetilde{y}_n^*\| \leqslant 6$. Let us show that $g_n = e_n$, $g_n^* = e_n^* + \widetilde{y}_n^*$ form the biorthogonal system we need. Indeed, consider an arbitrary $x = e + y \in X$, $e \in c_0$, $y \in Y$. If $\|y\| \leqslant \frac{1}{3}\|x\|$, then $\|e\| \geqslant \frac{2}{3}\|x\|$ and

$$\begin{aligned}\sup_{n\in\mathbb{N}} |g_n^*(x)| &= \sup_{n\in\mathbb{N}} |e_n^*(e) + \widetilde{y}_n^*(y)| \\ &\geqslant \sup_{n\in\mathbb{N}} |e_n^*(e)| - \frac{1}{3}\|x\| = \|e\| - \frac{1}{3}\|x\| \geqslant \frac{1}{3}\|x\|.\end{aligned}$$

In the opposite case of being $\|y\| > \frac{1}{3}\|x\|$, we select a sequence of indices $n_1 < n_2 < \cdots$ such that $\left\{|\widetilde{y}_{n_k}^*(y)|\right\} \longrightarrow \|y\|$. Then

$$\begin{aligned}\sup_{n\in\mathbb{N}} |g_n^*(x)| &\geqslant \limsup_{k\to\infty} |e_{n_k}^*(e) + \widetilde{y}_{n_k}^*(y)| \\ &= \limsup_{k\to\infty} |\widetilde{y}_{n_k}^*(y)| = \|y\| > \frac{1}{3}\|x\|.\end{aligned}$$ □

Lemma 3.2. *Let X be a separable Banach space containing an isomorphic copy of c_0. Then there is an isomorphic embedding $T : X \longrightarrow \ell_\infty$ such that $T(X) \supset c_0$.*

Proof. Remark that if X contains an isomorphic copy of c_0, then X can be renormed equivalently to have an isometric copy of c_0. After this, take $\{(g_n, g_n^*)\}_{n\in\mathbb{N}}$ from Lemma 3.1 and let us define $T : X \longrightarrow \ell_\infty$ as follows:

$$T(x) = \{g_n^*(x)\}_{n\in\mathbb{N}} \in \ell_\infty \qquad (x \in X).$$

The inequality (1) guaranties that

$$\frac{1}{3}\|x\| \leqslant \|T(x)\| \leqslant 12\|x\| \text{ for all } x \in X, \tag{3}$$

and the image of g_n is the n-th unit vector of $c_0 \subset \ell_\infty$, so $T(X) \supset c_0$. □

To finish our arguments, we need to use a class of lush spaces which was also introduced in the aforementioned paper [5], the so-called C-rich subspaces of $C(K)$.

Definition 3.3. Let K be a compact Hausdorff space. A closed subspace X of $C(K)$ is said to be *C-rich* if for every nonempty open subset U of K and every $\varepsilon > 0$, there is a positive function h of norm 1 with support inside U such that the distance from h to X is less than ε.

Some examples and remarks about C-rich subspaces will be needed.

Remarks 3.4.

(a) Due to [5, Proposition 2.5], *if K is a perfect compact space, then every finite-codimensional subspace of $C(K)$ is C-rich and, in particular, lush.*

(b) *If one considers ℓ_∞ as $C(\beta\mathbb{N})$, then c_0 is C-rich in ℓ_∞.* Indeed, this follows easily from the fact that $\mathbb{N}$ is a dense subset of $\beta\mathbb{N}$ consisting of isolated points.

(c) *If $X \subset C(K)$ is C-rich, then every subspace $Y \subset C(K)$ containing X is C-rich.*

(d) In particular, *every subspace of ℓ_∞ containing c_0 is C-rich.*

(e) *Let K be an infinite compact set and X be a Banach space such that it is C-rich in $C(K)$. Then, X contains an isomorphic copy of c_0.* Indeed, we take a sequence of disjoint open sets $V_n \subset K$. Since X is C-rich in $C(K)$, for $\varepsilon > 0$ and $n \in \mathbb{N}$ we can find $f_n \in C(K)$ such that

$$\operatorname{supp}(f_n) \subset V_n, \quad f_n \geqslant 0, \quad \|f_n\| = 1, \quad \text{and} \quad \operatorname{dist}(f_n, X) \leqslant \frac{\varepsilon}{2^n}.$$

The sequence $\{f_n\}$ is a c_0-basic sequence in $C(K)$, and a perturbation argument gives us a basic sequence in X which is equivalent to $\{f_n\}$ and so, it spans an isomorphic copy of c_0.

We are now able to state the main result of the section which characterizes isomorphically the separable Banach spaces containing c_0.

Theorem 3.5. *For a separable infinite-dimensional Banach space X, the following conditions are equivalent:*

(i) *X contains an isomorphic copy of c_0,*
(ii) *X is isomorphic to a C-rich subspace of $\ell_\infty = C(\beta\mathbb{N})$,*
(iii) *X is isomorphic to a C-rich subspace of some $C(K)$.*

Proof. $(i) \Rightarrow (ii)$. Lemma 3.2 tells us that there is an isomorphic embedding $T : X \longrightarrow \ell_\infty$ such that $T(X) \supset c_0$. Then, $T(X)$ is a C-rich subspace of ℓ_∞ by Remark 3.4.d and X is isomorphic to $T(X)$. The implication $(ii) \Rightarrow (iii)$ is evident and $(iii) \Rightarrow (i)$ is shown in Remark 3.4.e. □

The following result is an evident consequence of the above theorem.

Corollary 3.6. *Every separable Banach space containing an isomorphic copy of c_0 can be equivalently renormed to be lush and, in particular, to have numerical index* 1.

As an easy consequence we obtain the following.

Corollary 3.7. *Any closed subspace of c_0 can be renormed to be lush and, in particular, to have numerical index* 1.

Proof. Let X be a closed subspace of c_0. If X is finite-dimensional, the result is clear. Otherwise, X contains an isomorphic copy of c_0 [1, Proposition 2.1.1] and the result follows from the above corollary. □

As far as we know, these are the first non-trivial sufficient conditions for renorming with numerical index 1 appearing in the literature, and they give a positive partial answer to Problem 21 of [9].

We finish the section with an example showing that the answer to Problem 22 of [9] (whether any Banach space such that X^* contains an isomorphic copy of ℓ_1 is renormable with numerical index 1) is negative.

Example 3.8. *There is a Banach space Y such that Y^* is isomorphic to ℓ_1 but Y does not admit an equivalent norm with numerical index* 1. Indeed, let us consider the real space Y given in [4] such that Y^* is isomorphic to ℓ_1 and Y has the Radon-Nikodým property. Then, Y is an infinite-dimensional real Banach space having the Radon-Nikodým property and it is also Asplund, so [12, Corollary 4] shows that it does not admit an equivalent norm with numerical index 1.

4. A reformulation of lushness

In this section we prove a useful reformulation of lushness which does not use slices. We use it to reduces lushness of a non-separable space to the lushness of "sufficiently many" separable subspaces and to study lushness of ultraproducts.

Theorem 4.1. *Let X be a Banach space and let $G \subset S_{X^*}$ be a norming rounded subset for X. Then, the following are equivalent:*

(i) *X is lush.*

$(ii)_{\mathbb{R}}$ *In the real case: for every $x, y \in S_X$ and $\varepsilon > 0$, there exist $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$ and $x_1, x_2 \in B_X$ such that*

$$\|x + x_1 + x_2\| > 3 - \varepsilon$$

and

$$\|y - (\lambda_1 x_1 - \lambda_2 x_2)\| < \varepsilon$$

$(ii)_{\mathbb{C}}$ *In the complex case: For every $x, y \in S_X$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist $\lambda_1, \dots, \lambda_n \geqslant 0$, $\sum_{k=1}^{n} \lambda_k = 1$ and $x_1, \dots, x_n \in B_X$ such that*

$$\left\| x + \sum_{k=1}^{n} x_k \right\| > n + 1 - \varepsilon \tag{4}$$

and

$$\left\| y - \sum_{k=1}^{n} \lambda_k \exp\left(\frac{2\pi i k}{n}\right) x_k \right\| < \varepsilon + \frac{2\pi}{n} \tag{5}$$

(*iii*) *For every $x, y \in S_X$ and for every $\varepsilon > 0$ there is $x^* \in G$ such that $x \in S = S(B_X, x^*, \varepsilon)$ and $\operatorname{dist}(y, \operatorname{aconv}(S)) < \varepsilon$ (i.e. x^* in the definition of lushness can be chosen from G).*

Proof. Let us start with the complex case.

(*i*) ⇒ (*ii*). Fix $x, y \in S_X$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Since X is a lush space, we may find $x^* \in S_{X^*}$, $\mu_j \in [0, 1]$, $j = 1, \dots, N$ with $\sum_{j=1}^{N} \mu_j = 1$, $\theta_j \in [0, 2\pi]$ and $y_j \in S(B_X, x^*, \varepsilon/n)$ satisfying

$$\operatorname{Re} x^*(x) > 1 - \varepsilon/n \qquad \text{and} \qquad \left\| y - \sum_{j=1}^{N} \mu_j \exp(i\theta_j) y_j \right\| < \varepsilon\,. \tag{6}$$

Taking into account that the points $\left\{\frac{2\pi k}{n} : k = 1, \dots, n\right\}$ form an $\frac{2\pi}{n}$-net of $[0, 2\pi]$ we can represent the set of indices $\{1, \dots, N\}$ as a disjoint union of sets A_k, $k = 1, \dots, n$ in such a way that

$$\left| \theta_j - \frac{2\pi k}{n} \right| \leqslant \frac{2\pi}{n} \qquad \text{for every} \qquad j \in A_k. \tag{7}$$

Let us show that

$$\lambda_k = \sum_{j \in A_k} \mu_j, \quad \text{and} \quad x_k = \frac{1}{\lambda_k} \sum_{j \in A_k} \mu_j y_j \quad \text{if} \quad A_k \neq \emptyset$$

and

$$\lambda_k = 0, \quad \text{and arbitrary} \quad x_k \in S(B_X, x^*, \varepsilon/n) \quad \text{if} \quad A_k = \emptyset$$

fulfill the desired conditions. Indeed, it is clear that $x_k \in S(B_X, x^*, \varepsilon/n)$, which ensures validity of (4). The remaining condition (5) follows from (6) and (7):

$$\begin{aligned}
\left\| y - \sum_{k=1}^{n} \lambda_k \exp\left(\frac{2\pi i k}{n}\right) x_k \right\| &= \left\| y - \sum_{k: A_k \neq \emptyset} \sum_{j \in A_k} \mu_j \exp\left(\frac{2\pi i k}{n}\right) y_j \right\| \\
&\leqslant \left\| y - \sum_{k: A_k \neq \emptyset} \sum_{j \in A_k} \mu_j \exp(i\theta_j) y_j \right\| + \frac{2\pi}{n} \\
&= \left\| y - \sum_{j=1}^{N} \mu_j \exp(i\theta_j) y_j \right\| + \frac{2\pi}{n} < \varepsilon + \frac{2\pi}{n}\,.
\end{aligned}$$

(*ii*) ⇒ (*iii*). For given $x, y \in S_X$ and $\varepsilon_1 > 0$, we apply (*ii*) with $\varepsilon = \varepsilon_1/2$ and n big enough to ensure that $\frac{2\pi}{n} < \varepsilon_1/2$. We get $\lambda_1, \dots, \lambda_n \geqslant 0$, $\sum_{k=1}^{n} \lambda_k = 1$ and $x_1, \dots, x_n \in B_X$ satisfying (4) and (5). The first of these conditions ensures the existence of $x^* \in G$ such that x and all the x_k belong to the slice $S(B_X, x^*, \varepsilon) \subset S(B_X, x^*, \varepsilon_1)$. The second condition shows that $\operatorname{dist}(y, \operatorname{aconv}(S(B_X, x^*, \varepsilon_1))) < \varepsilon_1$, which proves lushness of X.

The last implication $(iii) \Rightarrow (i)$ is evident.

In the real case, $S_{\mathbb{R}}$ consists just of two points 1 and -1 and parameters n and $\exp\left(\frac{2\pi i k}{n}\right)$ just disappear from the statement. The proof just follows the same lines as in the complex case. $\square$

The following is an application of the above characterization.

Theorem 4.2. *For a Banach space X the following two conditions are equivalent:*

(i) *X is lush,*
(ii) *Every separable subspace $E \subset X$ is contained in a separable lush subspace Y, $E \subset Y \subset X$.*

Proof. The implication $(ii) \Rightarrow (i)$ is immediate from the definition of lushness.

Let us prove only the more bulky complex case of $(i) \Rightarrow (ii)$. Let (k_n, j_n) be the standard "triangle" enumeration of all pairs of naturals. Remark that, under this enumeration, $k_n \leqslant n$ and $j_n \leqslant n$ for all $n \in \mathbb{N}$. Let us construct recurrently a sequence of separable subspaces $E_1 \subset E_2 \subset \ldots$, dense sequences $A_m = \{a(m,r) : r \in \mathbb{N}\} \subset S_{E_m}$ and $x_m, y_m \in S_X$ as follows: for $m = 1$ we take $E_1 = E$, let $A_1 = \{a(1,r) : r \in \mathbb{N}\}$ be a dense sequence in S_{E_1} such that every of its elements repeats in it infinitely many times, and take $x_1 = a(k_{k_1}, j_{k_1}) = a(1,1)$, $y_1 = a(k_{j_1}, j_{j_1}) = a(1,1)$. If E_m and A_m are already constructed, we consider $x_m = a(k_{k_m}, j_{k_m})$, $y_m = a(k_{j_m}, j_{j_m})$ and apply item (ii) of Theorem 4.1 to x_m, y_m with $\varepsilon_m = 1/m$ for $n = 2, \ldots, m$. We get $\lambda_{m,n,1}, \ldots, \lambda_{m,n,n} \geqslant 0$, $\sum_{k=1}^{n} \lambda_{m,n,k} = 1$ and $x_{m,n,1}, \ldots, x_{m,n,n} \in B_X$ such that

$$\left\| x_m + \sum_{k=1}^{n} x_{m,n,k} \right\| > n + 1 - \varepsilon_m$$

and

$$\left\| y_m - \sum_{k=1}^{n} \lambda_{m,n,k} \exp\left(\frac{2\pi i k}{n}\right) x_{m,n,k} \right\| < \varepsilon_m + \frac{2\pi}{n},$$

$n = 2, \ldots, m$. Define E_{m+1} as the linear span of E_m and of all the vectors $x_{m,n,s}$, $n = 2, \ldots, m$, $s = 1, 2, \ldots, n$. Select $A_{m+1} = \{a(m+1, r) : r \in \mathbb{N}\}$ as a dense sequence in $S_{E_{m+1}}$ such that every of its element repeats in it infinitely many times. Put $Y = \overline{\bigcup_{m \in \mathbb{N}} E_m}$. Under this construction Y is separable, contains E, and the pair (x_m, y_m) runs over all possible values of $(a(r,s), a(t,u))$, $r, s, t, u \in \mathbb{N}$. Indeed, for the last assertion we observe that the map $m \longmapsto (k_m, j_m)$ is bijective from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, so the map $(m,n) \longmapsto ((k_m, j_m), (k_n, j_n))$ is bijective from $\mathbb{N} \times \mathbb{N}$ to $\big((\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})\big)$.

Thus, the composition of these two maps $m \longmapsto ((k_{k_m}, j_{k_m}), (k_{j_m}, j_{j_m}))$ is bijective from $\mathbb{N}$ to $\big((\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{N})\big)$ and so

$$\begin{aligned}\{(x_m, y_m) \,:\, m\in\mathbb{N}\} &= \big\{(a(k_{k_m}, j_{k_m}), a(k_{j_m}, j_{j_m}))) \,:\, m\in\mathbb{N}\big\}\\ &= \big\{(a(i,j), a(k,l)) \,:\, (i,j,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}\big\}.\end{aligned}$$

Therefore, $\{(x_m, y_m) \,:\, m\in\mathbb{N}\}$ is dense in $S_Y\times S_Y$ and every of its elements repeats in the sequence infinitely many times. Due to our construction, this means that item (ii) of Theorem 4.1 takes place for arbitrary x, y from the fixed dense subset $\{(x_m, y_m) \,:\, m\in\mathbb{N}\}$ of $S_Y\times S_Y$ for all $n\in\mathbb{N}$ and with arbitrarily small epsilons. This means that Y is lush. □

Let us comment that, since for Asplund spaces lushness is equivalent to having numerical index 1 (Theorem 2.1), the above result improves [13, Teorema 3], where it is proved the following. Let X be an Asplund, weakly countably determined space with numerical index 1. Then, for every separable subspace Y of X, there is another separable subspace Z of X containing Y and such that $n(Z)=1$.

We finish the section with an application of Theorem 4.1 to ultraproducts. Let us recall the notion of (Banach) ultraproducts [8]. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We can consider the ℓ_∞-sum of the family, $[\oplus_{n\in\mathbb{N}}X_n]_{\ell_\infty}$, together with its closed subspace

$$N_{\mathcal{U}} = \left\{\{x_n\}_{n\in\mathbb{N}}\in[\oplus_{n\in\mathbb{N}}X_n]_{\ell_\infty} \;:\; \lim_{\mathcal{U}}\|x_n\| = 0\right\}.$$

The quotient space $(X_n)_{\mathcal{U}} = [\oplus_{n\in\mathbb{N}}X_n]_{\ell_\infty}/N_{\mathcal{U}}$ is called the *ultraproduct* of the family $\{X_n\}_{n\in\mathbb{N}}$ relative to the ultrafilter $\mathcal{U}$. Let $(x_n)_{\mathcal{U}}$ stand for the element of $(X_n)_{\mathcal{U}}$ containing a given family $\{x_n\}\in[\oplus_{n\in\mathbb{N}}X_n]_{\ell_\infty}$. It is easy to check that

$$\|(x_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}}\|x_n\|.$$

If all the X_n are equal to the same Banach space X, the ultraproduct of the family is called the $\mathcal{U}$*-ultrapower* of X and it is usually denoted by $X_{\mathcal{U}}$.

Our next aim is to show that lushness is inherited by ultraproducts. Actually, the following stronger version of lushness will be obtained for the ultraproducts.

Definition 4.3. A complex Banach space X is said to be *ultra-lush*, if for every $x, y\in S_X$ and $n\in\mathbb{N}$ there exist $\lambda_1\ldots\lambda_n\geqslant 0$, $\sum_{k=1}^n\lambda_k = 1$ and $x_1,\ldots,x_n\in B_X$ such that

$$\left\|x+\sum_{k=1}^n x_k\right\| = n+1 \qquad\text{and}\qquad \left\|y-\sum_{k=1}^n\lambda_k\exp\left(\frac{2\pi ik}{n}\right)x_k\right\|\leqslant\frac{2\pi}{n}.$$

A real Banach space X is said to be *ultra-lush*, if for every $x, y\in S_X$, there exist $\lambda_1,\lambda_2\geqslant 0$, $\lambda_1+\lambda_2=1$ and $x_1, x_2\in S_X$ such that

$$\|x+x_1+x_2\| = 3 \qquad\text{and}\qquad y=\lambda_1x_1-\lambda_2x_2.$$

As straightforward applications of Theorem 4.1 and the definition of ultraproducts, we get the following results.

Corollary 4.4. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of lush spaces and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then $E = (X_n)_{\mathcal{U}}$ is ultra-lush. Moreover, the ultraproduct of any sequence of Banach spaces is ultra-lush whenever it is lush.*

Corollary 4.5. *Let X be a Banach space and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then, the ultrapower $E = (X)_{\mathcal{U}}$ is lush if and only if X is lush.*

5. Unconditional sums of lush spaces

A norm $\|\cdot\|_a$ on $\mathbb{R}^n$ is said to be an *absolute norm* if

$$\|(a_1,\ldots,a_n)\|_a = \|(|a_1|,\ldots,|a_n|)\|_a \qquad \big(a_1,\ldots,a_n\in\mathbb{R}\big)$$

and $\|(1,0,\ldots,0)\|_a = \cdots = \|(0,\ldots,0,1)\|_a = 1$. If $E = (\mathbb{R}^n, \|\cdot\|_a)$ is a space with an absolute norm and $X_1,\ldots,X_n$ are Banach spaces, we write $X = \big[X_1\oplus X_2\oplus\ldots\oplus X_n\big]_E$ to denote the E-direct sum of $X_1,\ldots,X_n$, that is, $X = X_1\oplus\cdots\oplus X_n$ endowed with the norm

$$\|(x_1,\ldots,x_n)\| = \|(\|x_1\|,\ldots,\|x_n\|)\|_a$$

We will use the fact that absolute norms are nondecreasing and continuous in each variable. For background, we refer the reader to [3, § 21]. Easy examples of absolute norms are the ℓ_p-norms for $1\leqslant p\leqslant\infty$ leading to the ℓ_p-direct sums of Banach spaces.

Definition 5.1. Let $E = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space with an absolute norm. We say that E *respects lushness* if for every collection $X_1, X_2,\ldots,X_n$ of lush spaces their E-direct sum $X = \big[X_1\oplus X_2\oplus\ldots\oplus X_n\big]_E$ is lush.

Our aim in this section is to characterize those absolute norms which respect lushness.

Theorem 5.2. *A space $E = (\mathbb{R}^n, \|\cdot\|)$ respects lushness if and only if it is lush itself.*

Proof. The "only if" part is evident: if E respects lushness, we can take all $X_k = \mathbb{R}$ and get lushness of $E = \big[X_1\oplus X_2\oplus\ldots\oplus X_n\big]_E$.

Let us prove the "if" part. Let $X_1, X_2,\ldots,X_n$ be lush spaces, $X = \big[X_1\oplus X_2\oplus\ldots\oplus X_n\big]_E$. First of all, let us remark that if all X_k are finite-dimensional, by [17, Theorem 3.1] lushness is equivalent to the following property

$$|x^*(x)| = 1 \qquad \big(\text{for every } x\in\operatorname{ext}(B_X) \text{ and every } x^*\in\operatorname{ext}(B_{X^*})\big).$$

This property can be easily verified taking into account that

$$\operatorname{ext}(B_X) = \{(a_1x_1,\ldots,a_nx_n) : (a_1,\ldots,a_n)\in\operatorname{ext}(B_E),\ x_k\in\operatorname{ext}(B_{X_k})\},$$

the analogous description for $\operatorname{ext}(B_{X^*})$, and the fact that the sets of extreme points in B_E and in B_{E^*} admit rotation in each variable.

Now, let us turn to the general case where some of X_k are infinite-dimensional. By a small perturbation argument, it is sufficient to prove the property of the definition of lushness for a fixed $\varepsilon > 0$ and $x, y \in S_X$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ with non-zero x_k, y_k. Denote

$$\alpha_k = \|x_k\|, \quad \beta_k = \|y_k\|, \quad \widetilde{x}_k = \frac{x_k}{\alpha_k}, \quad \widetilde{y}_k = \frac{y_k}{\beta_k}, \quad \text{and} \quad \delta = \frac{\varepsilon}{n+3}.$$

By lushness of X_k, there is $x_k^* \in S_{X_k^*}$ such that

$$\widetilde{x}_k \in S_k = S(B_{X_k}, x_k^*, \delta) \quad \text{and} \quad \operatorname{dist}(\widetilde{y}_k, \operatorname{aconv} S_k) < \delta.$$

This means that there is a finite number of elements $u_{k,1}, u_{k,2}, \ldots, u_{k,m_k} \in S_k$ and a finite number of scalars $\lambda_{k,1}, \ldots, \lambda_{k,m_k}$ such that $\sum_{j=1}^{m_k} |\lambda_{k,j}| = 1$, and

$$\left\| \widetilde{y}_k - \sum_{j=1}^{m_k} \lambda_{k,j} u_{k,j} \right\| < \delta. \tag{8}$$

We embed X_k isometrically into some infinite-dimensional space $\widetilde{X}_k$ (if X_k is infinite-dimensional, then put $\widetilde{X}_k = X_k$), denote $\widetilde{X} = \big[\widetilde{X}_1 \oplus \widetilde{X}_2 \oplus \ldots \oplus \widetilde{X}_n\big]_E$ and select linearly independent vectors $e_{k,1}, \ldots, e_{k,m_k} \in S_{\widetilde{X}_k}$ in such a way that

$$\sum_{j=1}^{m_k} \|e_{k,j} - u_{k,j}\| < \delta. \tag{9}$$

We consider $Y_k = \operatorname{span}\{e_{k,j} \,:\, j = 1, \cdots, m_k\}$ and introduce a new norm p_k on Y_k in such a way that $B_{(Y_k, p_k)} = \operatorname{aconv}\big(\{e_{k,j} \,:\, j = 1, \ldots, m_k\}\big)$. Remark that

$$p_k(w) \geqslant \|w\| \quad \text{for every } w \in Y_k. \tag{10}$$

Remark also that the linear map $g_k : (Y_k, p_k) \longrightarrow X_k$ defined by its values on the basis $\{e_{k,j}\}$ as $g_k(e_{k,j}) = u_{k,j}$, satisfies

$$g_k\big(B_{(Y_k, p_k)}\big) = \operatorname{aconv}\big(\{u_{k,j} \,:\, j = 1, \ldots, m_k\}\big)$$

and

$$g_k\big(\operatorname{conv}(\{e_{k,j} \,:\, j = 1, \ldots m_k\})\big) = \operatorname{conv}\big(\{u_{k,j} \,:\, j = 1, \ldots, m_k\}\big).$$

Moreover, thanks to (9), $\|w_k - g_k(w_k)\| < \delta$ for every $w_k \in B_{(Y_k, p_k)}$.

We introduce one more auxiliary space $(Y, p) = \big[(Y_1, p_1) \oplus \cdots \oplus (Y_n, p_n)\big]_E$. As proved before, since (Y_k, p_k) is isometric to $\ell_1^{(m_k)}$ for every k, Y is a finite-dimensional lush space and, therefore, $n(Y) = 1$, implying by [11, Corollary 3.7] that Y is a CL-space. This means, by definition, that B_Y is the absolutely convex hull of every maximal convex subset (maximal face) of S_Y.

For every $w = (w_1, \dots, w_n) \in B_{(Y,p)}$ we denote $g(w) = (g_1(w_1), \dots, g_n(w_n))$. Then, we get that

$$\|w - g(w)\|_{\widetilde{X}} < \|(\delta, \dots, \delta)\|_E \leqslant n\,\delta \tag{11}$$

and

$$\|g(w)\|_X \leqslant p(w). \tag{12}$$

Consider $u = (\alpha_1 \widetilde{z}_1, \dots, \alpha_n \widetilde{z}_n)$, where

$$\widetilde{z}_k = \frac{1}{m_k} \sum_{j=1}^{m_k} e_{k,j}$$

and $v = (\beta_1 \widetilde{v}_1, \dots \beta_n \widetilde{v}_n)$, where

$$\widetilde{v}_k = \sum_{j=1}^{m_k} \lambda_{k,j}\, e_{k,j}.$$

Then $u, v \in S_Y$, so there is an $f = (f_1, \dots, f_n) \in S_{(Y,p)^*}$ such that $f(u) = 1$ and the absolute convex hull of the face $F = \{w \in S_{(Y,p)} : \operatorname{Re} f(w) = 1\}$ contains v.

We denote $\gamma_k = p_k^*(f_k)$ and $D = \{k : \gamma_k \neq 0\}$. Then, we have $\gamma = (\gamma_1, \dots, \gamma_n) \in S_{E^*}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in S_E$ and

$$1 = \operatorname{Re} f(u) = \sum_{k=1}^{n} \alpha_k \operatorname{Re} f_k(\widetilde{z}_k) \leqslant \sum_{k=1}^{n} \alpha_k \gamma_k \leqslant 1.$$

Therefore,

$$\sum_{k=1}^{n} \alpha_k \gamma_k = 1, \quad \text{and} \quad f_k(\widetilde{z}_k) = \gamma_k \quad (k = 1, \dots, n). \tag{13}$$

So, in particular, $f_k/\gamma_k \in S_{(Y_k^*, p_k^*)}$ is a supporting functional of the point $\widetilde{z}_k$ for every $k \in D$, i.e. $f_k(e_{k,j}) = \gamma_k$ for all k, j.

Finally, let us introduce $x^* = (\gamma_1 x_1^*, \dots, \gamma_n x_n^*) \in S_{X^*}$ and let us prove that the slice $S = S(B_X, x^*, \varepsilon)$ is the one we need, namely that $x \in S$ and $\operatorname{dist}\big(y, \operatorname{aconv}(S)\big) < \varepsilon$.

The inclusion $x \in S$ is simple:

$$\operatorname{Re} x^*(x) = \sum_{k=1}^{n} \alpha_k\, \gamma_k\ \operatorname{Re} x_k^*(\widetilde{x}_k) > \sum_{k=1}^{n} \alpha_k\, \gamma_k\, (1 - \delta) = 1 - \delta.$$

To estimate the distance from y to $\operatorname{aconv}(S)$ we need the following claim:

Claim. $g(w) \in S$ for every element $w = (w_1, \dots, w_n) \in F$.

Proof of the claim. First of all, according to (12), $\|g(w)\| \leqslant 1$, so the only thing we need to prove is the estimation

$$\operatorname{Re} x^*(g(w)) > 1 - \varepsilon.$$

Denote $\phi_k = p_k(w_k)$. By the definition of F, we have

$$1 = f(w) = \sum_{k=1}^{n} \operatorname{Re} f_k(w_k) \leqslant \sum_{k=1}^{n} \gamma_k \phi_k \leqslant 1,$$

and, therefore,

$$\sum_{k=1}^{n} \gamma_k \phi_k = 1, \quad \text{and} \quad f_k(w_k) = \gamma_k \phi_k \ (k = 1, \dots, n). \tag{14}$$

This means that $w_k/\phi_k \in \operatorname{conv}\big(\{e_{k,j} \ : \ j = 1, \dots, m_k\}\big)$ for every k such that $\gamma_k \phi_k \neq 0$ and so, $g_k(w_k)/\phi_k \in \operatorname{conv}\big(\{u_{k,j} \ : \ j = 1, \dots, m_k\}\big) \subset S_k$. Therefore

$$\operatorname{Re} x^*(g(w)) = \sum_{k=1}^{n} \gamma_k \operatorname{Re} x_k^*(g_k(w_k)) \geqslant \sum_{k=1}^{n} \gamma_k \phi_k (1 - \delta) > 1 - \varepsilon,$$

which gives us the claim.

Now, let $u_1, u_2, \dots, u_m$ be elements of the face F and let $\lambda_1, \dots, \lambda_m$ be scalars such that $\sum_{j=1}^{m} |\lambda_j| = 1$, and $v = \sum_{j=1}^{m} \lambda_j u_j$. Thanks to the claim, $g(u_j) \in S$. Therefore

$$\operatorname{dist}\big(y, \operatorname{aconv}(S)\big) \leqslant \left\| y - \sum_{j=1}^{m} \lambda_j g(u_j) \right\| \leqslant \|y - v\| + \left\| v - \sum_{j=1}^{m} \lambda_j g(u_j) \right\|.$$

We continue the estimation using the definition of v and the inequalities (8) and (11):

$$\leqslant 2\delta \|(\beta_1, \dots, \beta_n)\|_E + \sum_{j=1}^{m} |\lambda_j| \|u_j - g(u_j)\| \leqslant (n+2)\delta < \varepsilon.$$

This completes the proof of the theorem. □

Although the above theorem only deals with finite sums of lush spaces, on can deduce from it the lushness of some infinite sums. Given an arbitrary family $\{X_i \ : \ i \in I\}$ of Banach spaces, we denote by $[\oplus_{i\in I} X_i]_{c_0}$ (resp. $[\oplus_{i\in I} X_i]_{\ell_1}$, $[\oplus_{i\in I} X_i]_{\ell_\infty}$) the c_0-sum (resp. ℓ_1-sum, ℓ_∞-sum) of the family.

Proposition 5.3. *Let $\{X_i \ : \ i \in I\}$ be a family of lush spaces. Then the c_0-, ℓ_1- and ℓ_∞-sums of the family are also lush.*

Proof. Let us start with the easier cases c_0- or ℓ_1-sum of the family. In these two cases, we consider the family of all finite sums (c_0- or ℓ_1 respectively) of elements of the family $\{X_i \ : \ i \in I\}$, viewed as subspaces of the whole c_0- or ℓ_1-sum. Then, Theorem 5.2 gives us that this is a family of lush subspaces which is filtered and whose union is dense. The result now follows straightforwardly from the definition of lushness.

For the ℓ_∞-sum of the family the above argument does not apply, but it is not difficult to give a direct proof. Let X denote the ℓ_∞-sum of the family, let (x_i), (y_i) be elements in the unit sphere of X and let $\varepsilon > 0$ be fixed. By the definition of X, there is $i_0 \in I$ such that $\|x_{i_0}\| > 1 - \varepsilon$. Since X_{i_0} is lush, we may find $x^*_{i_0} \in S_{X^*_{i_0}}$ such that, writing $S_{i_0} = S(B_{X_{i_0}}, x^*_{i_0}, \varepsilon)$, we have

$$x_{i_0} \in S_{i_0} \qquad \text{and} \qquad \operatorname{dist}\big(y_{i_0}, \operatorname{aconv}(S_{i_0})\big) < \varepsilon.$$

Now, defining $\widetilde{x}^* \in S_{X^*}$ as

$$\widetilde{x}^*\big((z_i)\big) = x^*_{i_0}(z_{i_0}) \qquad \big((z_i) \in X\big),$$

and writing $\widetilde{S} = S(B_X, \widetilde{x}^*, \varepsilon)$, we clearly have

$$(x_i) \in \widetilde{S} \qquad \text{and} \qquad \operatorname{dist}\big((y_i),\ \operatorname{aconv}(\widetilde{S})\big) < \varepsilon. \qquad \square$$

Let us comment that the analogue of Proposition 5.3 for Banach spaces with numerical index 1 was given in [16, Corollary 4]. On the other hand, we do not know of any analogous result to Theorem 5.2 for Banach spaces with numerical index 1.

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(Boyko and Kadets) Department of Mechanics and Mathematics, Kharkov National University, pl. Svobody 4, 61077 Kharkov, Ukraine

E-mail address: k_boyko@ukr.net, vova1kadets@yahoo.com

(Martín and Merí) Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

E-mail address: mmartins@ugr.es, jmeri@ugr.es