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The theory of representations as viewed from the onto-semiotic approach to mathematics education

Juan D. Godino* & Vicenç Font**

***Facultad de Educación, Universidad de Granada**
**** Facultat de Formació del Professorat, Universitat de Barcelona**
España

ABSTRACT: *The theory of representations set out by Goldin illustrates how the notion of representation is a potentially key element of a unifying theoretical framework for research into mathematics education. Systems of representation and the development of representational structures during mathematics learning and problem solving are considered as essential components of this framework. This paper looks at how Goldin conceives of representations and discusses their different types and connections with other notions; their strengths and certain limitations are also considered. Using the onto-semiotic approach (OSA) to mathematical knowledge we show that the notion of semiotic function and the categories of mathematical objects and processes used in the OSA complement the notion of representation, underlining, therefore, the importance of these notions as instruments for analysing mathematical thought and activity.*

Key words: theoretical frameworks**,** representation, mathematical objects and processes, meaning, teaching, learning

1. INTRODUCTION

Gerald Goldin presents the notion of 'system of representation' and its various types as the key concept of a unified psychological model of mathematical learning and problem solving (Goldin, 1998; 2008). He suggests that advances in the fields of psychology, formal linguistics, semantics and semiotics, together with the study of mathematical structures and the practical need to understand the interactions between students and computer-based environments, have led to intense work on representations and symbol systems in the psychology of mathematics education.

The interest shown by Goldin in several papers in which he develops a unified model of mathematics cognition and discusses its use in research on mathematics education is shared by the onto-semiotic approach (OSA) to mathematical knowledge and teaching (Godino & Batanero, 1998; Godino, 2002; Godino, Batanero & Font, 2007). In both theoretical frameworks the notion of representation plays a key role, although in the case of the OSA this notion is conceived of in terms of 'semiotic function' (Hjemslev, 1943; Eco, 1995) and is complemented by a typology of mathematical objects and processes, thus enabling more nuanced analyses to be conducted into how mathematical concepts are developed and communicated.

In this paper we analyse and study the points of agreement and complementarity between the OSA and the following aspects of the theory of representations, as described by Goldin in various articles (Goldin, 1998; 2008):

- External and internal representation
- Notion of representational system, components and structure
- Conventional nature and ambiguity

The paper begins by outlining the main notions of the onto-semiotic approach that will be used to interpret the theory of representations and to study the points of agreement and complementarity between the two frameworks. This study extends and complements our previous research into representations (Font, Godino & D'Amore, 2007; Font, Godino & Contreras, 2008).

2. THE ONTO-SEMIOTIC APPROACH TO MATHEMATICAL KNOWLEDGE

The onto-semiotic approach to mathematical cognition tackles the problem of meaning and the representation of knowledge by elaborating an explicit mathematical ontology based on anthropological (Bloor, 1983; Chevallard, 1992), semiotic and socio-cultural theoretical frameworks (Ernest, 1998; Presmeg, 1998a; Radford, 2006; Sfard, 2000). It assumes a certain socio-epistemic relativity (Cantoral, Farfán, Lezama & Martínez-Sierra, 2006) for mathematical knowledge, since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the institutions and the social context of which it forms a part (Radford, 1997).

Figure 1 shows some of the different theoretical notions of the onto-semiotic approach to mathematical knowledge. Here, mathematical activity plays a central role and is modelled in terms of systems of operative and discursive practices. From these practices, the different types of mathematical objects (problems, languages, concepts, propositions, procedures and arguments) emerge; these objects are interrelated, forming cognitive or epistemic configurations (hexagon in Figure 1). Lastly, the objects that appear in mathematical practices and those emerging from these practices, depending on the language game in which they participate (Wittgenstein, 1953), might be considered from the five facets of dual dimensions (decagon in Figure 1): personal/institutional, unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. The dualities, as well as the objects can be analysed from a process-product perspective, which leads us to the processes in Figure 1.

The onto-semiotic approach does not aim, at the outset, to provide a definition of "process", as there are many different types of processes: one can talk of process as a sequence of practices, as cognitive processes, meta-cognitive processes, processes of instruction, processes of change, social processes, etc. These are very different processes, and perhaps the only characteristic many of them have in common is the consideration of the *time* factor and, to a lesser degree, *the sequence in which each member takes part in the determination of the following*. Instead of giving a general definition of the process, therefore, the onto-semiotic approach opts to select a list of processes considered important in mathematical activity (those in Figure 1), without claiming that this includes all the processes implicit in mathematical activity, nor necessarily the most important ones; this is because some of the most important processes (for example, the process of understanding, the solving of problems or modelling) are more than processes: they are hyper or mega-processes.

The six types of postulated primary objects widen the traditional distinction between conceptual and procedural entities, which we consider insufficient for describing the objects that intervene in and emerge from mathematical activity. Problem-situations promote and contextualise the activity; languages (symbols, notations, graphics, etc.) represent the other entities and serve as tools for action; arguments justify the procedures and properties that relate the concepts. These entities have to be considered as functional and relative to the language game (institutional frameworks and use contexts) in which they participate; they also have a recursive character, in the sense that each object may be composed of other entities, depending on the analysis level—for example, arguments, which may involve concepts, properties, operations, etc. The concept, included as a component of the onto-semiotic configurations, is conceived of as "concept – definition"; this view is different from that of Vergnaud (1990), who conceives of a concept as the system formed by the “situations, operative invariants and representations”.

Meaning is not only a key notion in mathematics education but also in the onto-semiotic approach, where it is basically conceived of in two different ways that will be described later. In one of them meaning is interpreted in a very simple, powerful and operative way using the notion of “semiotic function” (Eco, 1978; Hjelmslev, 1943/1963): *Meaning is the content of any semiotic function*, that is to say, the content of the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to distinct criteria or a corresponding code. The content of the semiotic functions, and hence the meaning, could be a personal or institutional, unitary or systemic, ostensive or non-ostensive object; it could be a concept–definition, problem–situation, procedure, argument, or a linguistic element. In agreement with Peirce’s semiotics, the onto-semiotic approach also assumes that both the expression (antecedent of a semiotic function) and content (consequent) might be any type of entity.

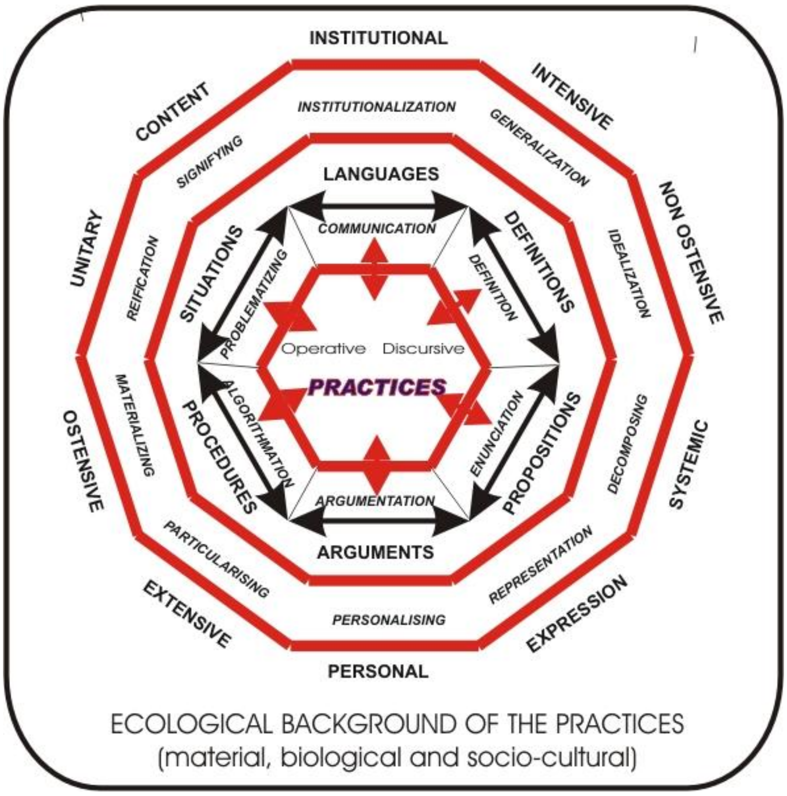

Figure 1: Configuration of objects and processes

3. THEORY OF REPRESENTATIONS: POINTS OF AGREEMENT AND COMPLEMENTARITY WITH THE OSA

Our analysis focuses on the following aspects of Goldin's theory of representations: external and internal representations, representational systems (components and structure), conventional nature and ambiguity.

3.1. External and internal representations

3.1.1 Interpretation within the framework of Goldin's theory of representations

External representations

The duality *internal/external* is a key notion for Goldin. External systems of representations comprise the conventional symbol systems of mathematics, such as base-ten numeration, formal algebraic notation, the real number line or Cartesian coordinate representation. Learning environments are also included, for example, those that use concrete manipulative materials or computer-based micro-worlds.

A representation is considered as a sign or configuration of signs, characters or objects that can stand for something else (to symbolise, code, provide an image of, or represent). The represented object can vary according to the context or the use of the representation: a Cartesian graph, for example, can represent a function or the solution set of an algebraic equation.

Some external systems of representation are mainly notational and formal, examples being systems of numeration, the writing of algebraic expressions, agreements about the expression of functions, derivatives, integrals, programming languages, etc. Other external systems show relationships visually or graphically, such as number lines, graphs based on Cartesian or polar coordinates, and geometric diagrams; the words and expressions of ordinary language are also external representations. These representations may denote and describe material objects, physical properties, actions and relationships, or objects that are much more abstract (Goldin, 1998, p. 4).

Internal representations

Internal representations comprise students' personal symbolisation constructs and assignments of meaning to mathematical notations. Under this category Goldin also includes students' natural language, their visual imagery and spatial representation, their problem-solving strategies and heuristics, and their affect in relation to mathematics. Internal cognitive configurations may or may not have structural similarities with external systems, at least in the unified model proposed by Goldin (1998, p. 147); the symbolic relationship can be established with external systems or between internal ones.

Internal cognitive (or mental) representations are introduced as a theoretical tool to characterise the complex cognitions which students may construct from external representations. They cannot be directly observed, but rather are inferred from observable behaviour.

As types of cognitive representations Goldin (1998) describes the following:

- Verbal or syntactic: capacities related to the use of natural language by individuals, mathematical and non-mathematical vocabulary, including the use of grammar and syntax.
- Figural (imagistic) and gestural systems, including spatial and visual cognitive configurations, or 'mental images'; gestural and body schema.
- Mental manipulation of formal notations (numerals, arithmetic operations, visualisation of symbolic steps to solve an equation)
- Strategic and heuristic processes: 'trial and error', 'breakdown into stages', etc.
- Affective systems of representation: emotions, attitudes, beliefs and values with respect to mathematics, or about themselves in relation to mathematics.

Interaction between external and internal representations

The interaction between external and internal representations is considered to be fundamental for teaching and learning. The primary concern of the teaching process centres on the nature of the internal representations being developed by students. The connections between representations can be based on the use of analogies, images and metaphors, as well as on structural similarities and differences between systems of representation. Internal representations are always inferred from their interactions with — or discourse about — the production of external representation. It is useful to consider that the external represents the internal, and vice-versa. A mathematical concept has been learnt and can be applied insofar as a variety of appropriate internal representations (and the functional relationships between them) have been developed.

3.1.2 Interpretation within the framework of the OSA

The classification into mental or internal representations and external representations is not by any means a transparent one. Indeed, the ambiguity of the internal/external classification has been pointed out by different researchers. The reason for this is that mathematical objects are represented in books or on boards, etc. by means of mathematical sign systems and using materials that form part of the real world; given that the subject is also assumed to relate to this real world by means of mental representations it transpires that what is considered to be external is also, in a way, internal. For example, Kaput, in relation to this question, asks: "What is it [a mental representation]? What do we mean when we say it "represents" something? For whom? How? What is the difference between the experience of an internal representation and that of an external representation? And is an external representation a socially or a personally constituted system?" (Kaput, 1998, p. 267)

Furthermore, the classification between internal and external representations obliges us to ask what goes first: the internal before the external or vice-versa. The majority of cognitive psychologists regard internal representations as more basic since they consider that for the external representations to be representations they have to be mentally represented by their users and, furthermore, the mental representations can exist without a public duplicate; for example, many of our memories are never communicated. In contrast, the majority of social scientists and many philosophers inspired by Wittgenstein do not agree with this.

We consider that the internal/external duality does not explain the institutional dimension of mathematics knowledge, thus confusing to a certain extent the said objects with the ostensive resources that are used as a support for the creation or emergence of institutional entities. This has serious consequences in terms of understanding the learning process, since the role of human activity and social interaction are not adequately modelled in the production of mathematics knowledge and in learning.

In the onto-semiotic approach the internal/external classification, in addition to being problematic, is considered as not very operative, and so it is proposed to convert it into two dualities or contextual attributes that, in our opinion, are more useful. These dualities are ostensive/non-ostensive and personal/institutional.

Ostensive/non-ostensive

In mathematical practices one form which mathematical objects can take is related to the duality *ostensive/non-ostensive*. These two forms must be considered in an intersubjective sense: "something" can be shown directly to another, versus "something" cannot itself be shown directly, but only by means of another "something" that can be shown directly. To put it another way, mathematical ostensives have a characteristic akin to things like oranges, tables, etc., i.e. they have real existence in time and space. In contrast, this kind of existence is not attributed to non-ostensive objects, which are usually considered to have an ideal existence.

Contrary to the usual view the OSA, in line with our conceptualisation of mathematical objects, considers as mathematical a type of object that is ostensive, i.e. objects that have a material existence. In this category, one must consider the material representations of non-ostensive objects (for example, written expressions or oral and gestural discourse), as well as material examples of mathematical concepts considered as extra-mathematical (for example, three oranges, or the circumference drawn on the board). We also believe it useful to include in this category the so-called "instruments" that also participate in mathematical practices (for example, the compass).

Personal/institutional

In the OSA it is considered that mathematical cognition must contemplate the personal and institutional facets, between which complex dialectical relationships are established, and whose study is essential for mathematics education. 'Personal cognition' is the result of the individual subject's thought and action with respect to a certain class of problems, whereas 'institutional cognition' is the result of the dialogue, agreement and rules that emerge from a group of individuals who form a practice community.

Another form that mathematical objects can take in mathematical practices is related to the duality *personal/institutional*. They may participate as personal or institutional objects and, depending on the language game, may shift from being personal to institutional. The personal/institutional dialectic is essential in teaching processes, whose aim is to enable students to take on board the relevant institutional objects.

3.2. Systems of representation, components and structure

3.2.1 Interpretation within the framework of Goldin's theory of representations

The notion of representation proposed by Goldin (2008) is certainly very general, as illustrated by the following passage:

> "The nature of representing relation between the one configuration and the other must eventually be made explicit…. The representing configuration might, for instance, act in place of, be interpreted as, connect to, correspond to, denote, depict, embody, evoke, label, link with, mean, produce, refer to, resemble, serve as a metaphor for, signify, stand for, substitute for, suggest, or symbolise the represented done. It might do one (or more) of these things by means of a physical linkage or a biochemical, mechanical, or electrical production process, or in the thinking of an individual teacher or student, or by virtue of the explicitly-agreed conventions or the tacitly-agreed practices of a social group or culture, or according to a model developed by an observer" (p. 179)

With respect to the teaching and learning of mathematics the term ‘representation’ and the expression ‘system of representation’ have the following interpretations (Goldin & Janvier, 1998, p. 1):

> “1. An external, structured physical situation, or structured set of situations in the physical environment, that can be described mathematically or seen as embodying mathematical ideas;
>
> 2. A linguistic embodiment, or a system of language, where a problem is posed or mathematics is discussed, with emphasis on syntactic and semantic structural characteristics;
>
> 3. A formal mathematical construct, or a system of construct, that can represent situations through symbols or through a system of symbols, usually obeying certain axioms or conforming to precise definitions – including mathematical constructs that may represent aspects of other mathematical constructs;
>
> 4. An internal, individual cognitive configuration, or a complex system of such configurations, inferred from behaviour or introspection, describing some aspects of the processes of mathematical thinking and problem solving.”

The relationship of representation (symbolisation or coding) between two systems is reversible. Depending on the context a graph may provide a geometric representation of a two-variable equation, or alternatively an equation ($x^2 + y^2 = 1$) may offer an algebraic symbolisation of a Cartesian graph.

Mathematical representations cannot be understood in isolation. An equation or a specific formula, a certain arrangement of multibase blocks, or a particular graph in a Cartesian system only acquire meaning as part of a wider *system* with established meanings and conventions.

> “The representational system important to mathematics and its learning have structure, so that different representations within a system are richly related to one another” (Goldin & Shteingold, 2001, pp. 1-2)

Each representational system includes the conventions that configure it, as well as the relationships with other mathematical objects and systems. In order to interpret the numeral 12, for example, it is necessary to incorporate the rules of the decimal positional numbering system and all the relationships this maintains with other numbering systems and with all the system of real numbers.

3.2.2 Interpretation within the framework of the OSA

The processes of representation and meaning associated with the duality expression/content (brought into correspondence by semiotic functions) correspond to the notion of representation in Goldin's theory, while the configuration of objects and processes extends that of the system of representation. The reversible nature of the representation is a feature shared by the semiotic function: the antecedent and consequent terms can be any type of object.

Relationship between configurations and systems of representation

The different interpretations of systems of representation proposed by Goldin & Janvier (1998, p. 1) are some of the elements considered in the configuration of objects and processes which are activated and emerge in mathematical practices (Figure 1). Thus:

> 1. An external, structured physical situation, or structured set of situations in the physical environment, that can be described mathematically or seen as embodying mathematical ideas;

This is the component referred to as 'problem situation' in Figure 1, seen in terms of its ostensive and extensive (particular) nature.

> 2. A linguistic embodiment, or a system of language, where a problem is posed or mathematics is discussed, with emphasis on syntactic and semantic structural characteristics.

In our case this is the 'language' component in Figure 1, seen in terms of its ostensive and content-related (meaning) nature.

> 3. A formal mathematical construct, or a system of constructs, that can represent situations through symbols or through a system of symbols, usually obeying certain axioms or conforming to precise definitions – including mathematical constructs that may represent aspects of other mathematical constructs.

In our model these are the remaining primary entities (definitions, propositions, procedures and arguments).

> 4. An internal, individual cognitive configuration, or a complex system of such configurations, inferred from behaviour or introspection, describing some aspects of the processes of mathematical thinking and problem solving

In our case these are the configurations of objects and processes seen from the personal perspective.

Figure 1 helps us provide a definition of "system of representation" within the framework of the OSA, one which we consider to be operative and suitable for the analysis of teaching and learning processes: it is the system formed by the configuration of objects that intervene in and emerge from a system of practices, together with the processes of signification and representation which are established between them (i.e. including the chain of semiotic functions that relates the objects which constitute the configuration).

In other words, carrying out a mathematical practice requires the activation of a configuration of objects (situations/problems, languages, procedures, definitions, propositions and arguments). These primary objects may be participating in mathematical practices as representations or as represented objects, and, depending on the language game, they may shift from being representations to being represented objects.

Figure 1 also illustrates some of the limitations of the 'system of representation' construct when it comes to analysing the inherent complexity of mathematical activity, since this construct does not take into account some of the aspects that feature in this figure and which are necessary for understanding mathematical activity; they may even be necessary for understanding part of this activity, as in the case of processes of representation and signification.

To put it another way, in addition to contemplating the inner part of Figure 1 from the perspective of the 'expression/content' duality, which leads us to the idea of system of representation, it is helpful to 'contemplate' not only the practices and object configurations activated in deriving it (inside the hexagon) but also the expression/content duality itself from the perspectives offered by the other dualities.

The processes of representation and signification associated with the expression/content duality can be 'contemplated' using some of the perspectives offered by Figure 1. Thus, we will first locate the process of interest (either that of representation or meaning) in the centre of the hexagon so as to relate this process with the processes of communication, enunciation, definition, argumentation and algorithmisation. Next we will locate it in the centre of the decagon in order to analyse it from the different perspectives offered by the dual facets. Finally, and for reasons of space, we will only apply the different perspectives offered by the dual facets to, primarily, the process of representation and, to a lesser extent, that of signification.

Representation in relation to the duality 'extensive/intensive'

The process of representation is related to the processes of particularisation and generalisation that are associated with the extensive/intensive duality. We constantly seek to break reality down, in some way or another, into a multiplicity of identifiable and discriminable objects, to which we refer by means of singular and general terms (this chair, a table, etc.). This also occurs when we analyse mathematical practices (the letter *x* on the board, the function f(x) = 3x + 2, etc.). The extensive/intensive facet (exemplar/type; particular/general) acts upon these objects.

Another form which mathematical objects can take in mathematical practices is thus related to the extensive/intensive duality. They may be participating as particular or general objects and, depending on the language game, they may shift from being particular to general or vice-versa.

Mathematical reasoning, going from the general to the general, introduces an intermediate phase that consists in contemplating a particular object. However, this fact poses a serious dilemma: the reasoning has to be carried out on a specific object (for example, a triangle), but this object refers to a generalisation. Furthermore, since the specific object is linked to its representation there is also the problem of whether the representation refers to a specific object or a general concept (Font, Godino & D'Amore, 2007).

The introduction of the extensive/intensive duality in the onto-semiotic approach can help to clarify the problem of the use of generic elements (Font & Contreras, 2008). This duality becomes an essential instrument in analysing the complexity associated with the use of the generic element. Expressed differently, the use of the generic element is associated with a complex network of semiotic functions (and, therefore, representations) that relate intensive with extensive objects. What these semiotic functions have in common is that they are all of the representational type, in the sense that they facilitate the representation of the expression of content; however, they can also be of different types according to whether the expression or the content are extensive or intensive, and according to the correspondence criterion between the expression and the content. For example, when a subject relates an object with the class to which it belongs, this semiotic function is, on one hand, representational, in the sense that the particular case can be taken as a representative of the class. However, on the other hand, it is of the metonymic type (in particular, a synecdoche [Presmeg, 1998b]) since an extensive (a part) is taken for the whole (the class); in this case the correspondence criterion is that of "belonging". In some cases the semiotic functions are exclusively representational.

Representation in relation to the duality 'expression/content'

In order to apply the facet 'extensive/intensive' to objects, these require a sign that enunciates them (in a certain way, it accompanies them). Although small children may not differentiate between sign and object, adults do make this distinction. Peirce (1978, §2.228) mentions this idea:

> "A sign, or representamen, is something which stands to somebody for something in some respect or capacity."

Given that the 'sign' and the 'object' are 'something', one must remember that they are both objects. Being an object or a sign is relative, and thus it is useful to distinguish between objects and signs. However, although this distinction is important it is one based on a temporary relationship rather than substance, since what is a sign at one moment may become an object at another, or vice-versa. Although in the 'non-differentiation' stage the subject identifies (confuses) the sign with the object, in the 'differentiation' stage the subject is able, as required, to make this identification or differentiate between sign and object.

> "To stand for, that is, to be in such a relation to another that for certain purposes it is treated by some mind as if it were that other. Thus, a spokesman, deputy, attorney, agent, vicar, diagram, symptom, counter, description, concept, premise, testimony, all represent something else, in their several ways, to minds who consider them in that way." (Charles S. Peirce, 2.273, The Collected Papers).

The possibility of differentiating between sign and object enables 'somebody' to establish a dyadic relationship between 'two objects' ('something' for 'something'). In this relationship ('something' for 'something') it is normally considered that one of the objects (the sign) is an 'expression' which is related with a 'content' (the other object).

The OSA considers that mathematical activity and the processes of constructing and using mathematical objects are characterised by being essentially relational. The different objects must not be conceived of as isolated entities, but rather as being in a relationship to one another. The distinction between expression and content enables us to take into account the essentially relational nature of mathematical activity. The relation is established by means of semiotic functions, understood as a relationship between an antecedent (expression) and a consequent (content) term that is established by a subject (person or institution) in accordance with a certain criterion or code of correspondence.

Godino & Batanero (1998) consider a semiotic function, at least metaphorically, as a correspondence between sets that brings into play three components: a plane of expression (initial object); a plane of content (final object); and a criterion or correspondence rule. The initial and final objects are constituted by any mathematical object. In the OSA, semiotic functions relate two objects that may be material or mental. This way of understanding semiotic functions is inspired by a long tradition that stretches from Peirce to Schütz by way of Husserl. The interpretation of semiotic functions as proposed by the OSA leads to a radical generalisation of the notion of representation that is used in cognitive research into mathematics education.

Representation in relation to the dualities 'ostensive/non-ostensive' and 'personal/ institutional'

If we formulate the question, *How is the sign related to the object?,* we come up against the problem of the classification of internal versus external representations that was discussed in the previous section. For example, if a subject sees written the word 'clock' and points to a clock on the wall, it is considered that the written sign 'clock' is related to the physical object by means of the concept (interpretant) which the subject (interpreter) has of it. For instance, the subject may consider that the clock is an instrument which, by means of a uniform movement, measures and indicates the time.

Normally, both the subject's concept and the sign are regarded as representations. It is also considered that the written word 'clock' is an external representation and the concept an internal (mental) representation.

This initial classification into *mental or internal representations* and *external representations* is not in any way a transparent one, as pointed out in the previous section. In addition to being problematic the onto-semiotic approach also considers the internal/external classification to be not very operative, and therefore it proposes transforming it into two dualities or contextual attributes that are regarded as more useful. These dualities are ostensive/non-ostensive and personal/institutional.

There are several reasons why, in mathematical discourse, and whether implicitly or explicitly, a distinction is made between ostensive and non-ostensive objects. Here we will focus on just two, the first being that in mathematics discourse it is possible to talk about ostensive objects representing non-ostensive objects that do not exist. For example, we can say that $f'(a)$ does not exist because the graph of $f(x)$ has a pointed form in x = a (Font, Godino, Planas & Acevedo, in press). The existence of well-established ostensive objects that represent non-ostensive ones that do not exist facilitates the consideration of the non-ostensive object as different from the ostensive that represents it.

The second reason is that there are different representations which are regarded as representations of the same mathematical object. For example, Duval (1995, 2008) has pointed to the importance of the different representations and transformations between representations in students' understanding of the mathematical object as something different from its representation.

This refinement of the internal/external duality by means of the ostensive/non-ostensive and personal/institutional dualities is a theoretical tool that enables us to clarify and differentiate processes and objects which, despite being very different, are often presented simply in terms of the 'internal/external' classification. For example, the mathematical notion of derivative function would be considered as external according to the duality ‘internal/external’, as would be the symbol f’(x) written in ink in a mathematical text. Yet if we apply jointly the dualities ‘ostensive/non-ostensive’ and ‘personal/institutional’, then the former example would be regarded as a non-ostensive institutional object, while the latter would be an institutional ostensive object.

The dualities ‘ostensive/non-ostensive’ and ‘personal/institutional’ are useful for analysing the differences between different interpretations of the expression ‘system of representation’ as considered by Goldin & Janvier (1998, p. 1). In the first and second of these interpretations the ostensive perspective takes precedence, whereas in the third it would be the institutional and in the fourth the personal.

Representation in relation to the duality ‘unitary/systemic’

One way of conceiving of a word’s meaning is to consider it as the content that is associated with the said expression (semiotic function). This is an elemental or ‘unitary’ way of understanding meaning. Given a mathematical object, which is considered as expression, the meaning is the mathematical object considered as content. A prototypical example would be the semiotic function that associates a definition (content) with a term (expression).

Another possible way of tacking the problem of ‘meaning’ is to do so in terms of usage. From this perspective the meaning of a mathematical object must be understood in terms of what can be done with it. This is a ‘systemic’ perspective, as it considers the meaning of the object to be the set of practices in which said object plays a determining role (or not).

These two ways of understanding meaning complement one another, since mathematical practices involve the activation of configurations of objects and processes that are related by means of semiotic functions.

One need only take a historical perspective on any institutional, non-ostensive mathematical object to illustrate the complexity of relationships that are established between: (1) a mathematical object; (2) its associated ostensive objects; (3) the practices that enable these ostensive objects to be manipulated; and (4) the situations in which the object is used (together with its ostensive objects and associated practices) to organise phenomena. An example of such a study is that conducted by Font & Peraire (2001) for the cissoid. This historical approach also reveals how the different ostensive forms that may represent a mathematical object are the result of a long process in which, in some cases, a new form of representation leads to the expression of a new research programme. This fact has important implications, the three most significant, and which have also been pointed out by Goldin, being as follows:

(1) Ostensive representations cannot be understood in isolation, as pointed out by Goldin & Shteingold (2001, p. 2). Therefore, it is more useful to talk about systems of signs rather than ostensive representations or signs.

(2) The fact that the same object can fit within two different research programmes, each with its own systems of representation, means that 'each representation' can become a 'represented object' of the representation from the other research programme.

(3) An ostensive representation has, on the one hand, a representational value: it is something that can stand for something other than itself. However, on the other hand, it has an instrumental value: it enables us to carry out certain practices that would not be possible with another type of representation. The representational value leads us to understand representation in an elemental or unitary way ('something' for 'something'). In contrast, the instrumental value leads us to understand representation in a *systemic* way, as the 'tip of the iceberg' of a complex system of practices which are made possible by this representation.

In the onto-semiotic approach the introduction of the duality 'unitary/systemic' enables us to reformulate the 'naïve' view that 'there is a single object with different representations'. What there is, in fact, is a complex system of practices in which the 'non-ostensive, institutional object' does not appear, whereas what do appear are configurations of primary objects (hexagon in Figure 1) that, implicitly, are assumed to be properties, representations or definitions of the said object, each one of which makes possible a sub-set of practices that are considered as the object's meaning. In other words, the non-ostensive, institutional object, understood as emerging from a system of practices, can be considered as a single object with a holistic meaning. However, each sub-set of practices contains a different configuration (hexagon in Figure 1), in the sense that different practices are enabled. Systems of practices can be divided into different kinds of more specific practices, which are made possible by a given configuration of objects and processes, thus enabling a distinction to be made between sense and meaning (Font & Ramos, 2005): senses can be interpreted as partial meanings.

Moving from analysis in terms of representations to analysis in terms of onto-semiotic configurations (mathematical objects and processes linked to mathematical practices) is necessary to obtain a better understanding of the complexities of how mathematics is taught and learnt. Certainly, representation and interpretation processes are crucial, and should be the focus of attention at a first level of analysis. However, a socio-epistemic and cognitive analysis that sheds light on the conflicts in teaching and learning requires a systematic look at the diverse types of objects and processes involved in the activity. At a second level of analysis, we should focus on the configurations of primary objects (languages, problems, definitions, propositions, procedures and argumentations) and the related primary processes. The third level of socio-epistemic and cognitive analysis should be centred on the contextual attributes and secondary associated processes: personalisation – institutionalisation; particularisation – generalisation; materialisation – idealisation; reification – decomposition.

3.3. Conventional nature and ambiguity

3.3.1 Interpretation within the framework of Goldin's theory of representations

For Goldin, systems of representation are conventional constructs, in the same way that a system of mathematical axioms can be. The decision as to where a system begins and ends, or whether an additional structure is intrinsic to a given system or the result of a symbolic relationship between two systems, is an arbitrary one that depends on the appropriateness and simplicity of the description. "For instance, at the elementary school level, there is nothing "objectively true" about the fact that an expression such as 3+4×5 is evaluated by performing the multiplication before the addition, and not by performing the addition first. It is a matter of commonly agreed notation, open to inventive modification" (Goldin, 2008, p. 180). Once a mathematical system has been established, with its postulates and conventions chosen so as to be useful for describing our world, we must necessarily accept the 'existence' of certain types of patterns and the 'truth' of certain mathematical statements. "Having assumed the conventional properties of natural numbers, our base ten notational system, the conventional definition of addition and multiplication, and the conventional definition of a prime number, it is true that 23 is a prime while 35 is not" (Goldin, 2008, p. 180).

In a representational system there may a degree of ambiguity that affects the system's set of syntactic and semantic rules, in that there may be exceptions to these rules. Although mathematics is characterised by the clarity and precision of its concept definitions and propositional statements, as well as by the rigour of its proofs, a certain degree of ambiguity and flexibility in the use of systems of representation (languages, symbolic notations, etc.) may facilitate critical thought and heuristic processes. In practice, the ambiguity is resolved by taking into account the context in which the sign, the configuration or the ambiguous symbolic relationship appears.

3.3.2 Interpretation within the framework of the OSA

In the OSA, and in accordance with Wittgenstein's (1978) philosophy of mathematics, some primary objects of epistemic configurations (procedures, definitions and propositions) are conceived of as 'grammatical' rules of a certain kind. From this point of view, mathematical statements are (grammatical) rules regarding the use of a certain type of signs, because in fact they are used as rules; they do not describe properties of ideal mathematical objects, even if this may appear to be the case. This is a conventionalist position that is opposed to the realist argument in the philosophy of mathematics: mathematical statements do not describe any type of reality (neither ideal nor natural) that exists a priori of the mathematician's constructive activity.

The ambiguity of systems of representation is basically explained in the OSA through reference to the way in which the meaning of mathematical objects (non-ostensive and institutional) is understood, since the same object may have a wide variety of meanings. As pointed out in the previous section, this object may participate in many different practices, in which different configurations of objects and processes (different meanings) are activated. These practices are linked to problem situations and are relative to the context of use, language games or institutional frameworks. Moreover, in these practices, the primary objects in the configurations may be the expression or content of many different semiotic functions. In each case one will have to elucidate the corresponding meaning according to the context.

4. FINAL REFLECTIONS

In the theory of representations the notion of 'representation' is proposed as a key element of a unified theoretical framework for research into mathematics education: "We should draw the natural conclusion — that it is time to set aside dismissive epistemologies, in order to proceed with concepts that can unify the understandings reached from disparate perspectives" (Goldin, 2008, p. 197).

This objective of building a unified theoretical framework for research into mathematics education is also the basis of the onto-semiotic approach. The OSA considers semiotic function (in both its representational and operational sense) to be a key notion, although it must be complemented by an explicit ontology that focuses attention on the configurations of mathematical objects and processes that enter into relation with one another. The assumption within the OSA of pragmatic/anthropological premises that influence mathematical activity, as well as the adoption of a notion of mathematical object that is compatible with these premises (i.e. emerging from systems of practices), has enabled the approach to develop object categories and types of relationship that open up new avenues for analysing and explaining didactic phenomena.

The didactic analysis of teaching and learning processes in mathematics involves not only the epistemic and cognitive dimensions, but also the affective, instructional and ecological dimensions. To date, the OSA has focused on the epistemic, cognitive and instructional dimensions, for which detailed categories of analysis have been developed (Godino, Contreras & Font, 2006). However, a system of categories for analysing the affective dimension and the interactions with the remaining dimensions has yet to be developed. In our view the model of affective representations developed by Goldin (Goldin, 2000; DeBelis & Goldin, 2006) could be articulated in a coherent way within the framework of the OSA. Developing this idea will be the subject of future studies.

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