# EPISTEMIC AND COGNITIVE ANALYSIS OF AN ARITHMETIC – ALGEBRAIC PROBLEM SOLUTION[1]

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**Abstract:**

*We analyze the solution of a problem proposed on the context of a mathematics methods course for in-training primary teachers, by applying some notions of the "onto-semiotic approach" to mathematics knowledge. The generalization of the arithmetic statement given to students involves the use of algebraic reasoning, giving way to the study of the relations between arithmetic and algebra, as well as the relations between empirical and deductive argumentation. The solution of elementary problems and the epistemic-cognitive reflection on the objects and meanings used during the solution is proposed as a means to overcome a limited students' conception on the nature of mathematics, usually reduced to its conceptual and procedural aspects.*

## 1. INTRODUCCTION

To train future primary teachers on mathematics and its didactics it is necessary to select problems whose solution put into effect competences on different subject matter knowledge fields (arithmetic, geometry, measure, stochastic, algebraic thinking) and that promote the articulation among the competences of mathematic and didactic type.

On the other hand, the design and implementation of the study processes ask for the teacher to develop analytical competences of the mathematic objects and processes used during the solution of the mathematic problems, so that the teacher can foresee conflicts on the meaning and on the many possibilities of institutionalization of the mathematic knowledge under consideration.

On this report we present and analyze a problem that requires the use of some arithmetic and algebraic content, and at the same time promote the reflection on the role of argumentative deduction, its relative effectiveness and validity compared to empirical verification.

This situation has been used in a class with in-training primary teachers and allows us to illustrate the kind of mathematic activities that we consider potentially useful for the teacher, due that it gives him/her the opportunity to think about his/her own way to conceive the mathematic activity. "The need for teachers to know mathematics differently than mathematicians do has been recognized by educator for a very long while" (Showder, 2007, p. 162).

[1] ICME 11, *Topic Study Group 27*, *Mathematical Knowledge for Teaching*. Monterrey, Mexico, 2008. http://tsg.icme11.org/document/get/391

We consider that this activity provides in-training teachers with a kind of mathematic experience in accordance to the principles proposed by Cooney and Wiegel (2003, p. 806): a pluralistic vision of the mathematics, reflection on the school mathematics and the style of teaching oriented towards the generating processes of mathematic knowledge. Furthermore, this kind of analytical activities helps teachers to realize the specific epistemological status of the students' mathematical knowledge. "The teacher has to be able to diagnose and analyze students' constructions of mathematical knowledge and has to compare those constructions to what was intended to be learned in order to vary the learning offers accordingly" (Steinbring, 1998, p. 159).

In a more specific way, we consider that the kind of analysis of the mathematic activity that we describe on this report is useful to illustrate some aspects of the teacher's activities:

- Design of didactic units centered on the resolution of problems, overcoming the conceptual-procedural dichotomy on the conception of mathematics and recognizing the key role of the representation and interpretation processes.

- Implementation and assessment of didactic units having in mind the student's cognitive configurations and foreseeing possible ways of institutionalization of the proposed knowledge.

We begin with the description of the context of the experience, the problem and the questions proposed to motivate the epistemic-cognitive reflection. In the next section we include the expected solution and the analysis of the mathematic objects and processes put into effect. The analysis of the answers offered by the students, by using the same tools proposed for the epistemic analysis, can contribute to develop competences to analyze the learning and comprehension reached by students.

## 2. EXPERIENCE CONTEXT AND FORMATIVE CICLE

The experience that we describe and the type of analysis that we carry on – by applying some notions of the "onto semiotic approach" of mathematic knowledge (Godino and Batanero, 1998; Godino, Batanero and Roa, 2005; Godino, Batanero and Font, 2007) - is part of a formative cycle on mathematics and its didactics intended to in-training mathematics' primary teachers, that includes the following steps:

1) Proposal and solution of a problem on elementary mathematics.

2) Reflection on the mathematic objects and processes put into effect during the solution of the problem.

3) Group discussion and institutionalization of the answers given by the students.

The given problem (section 2.1) was proposed to a group of 84 future teachers in the context of a mathematic methods course. The specific activity's objectives were:

- To create an introductory situation for the students to reflect on the properties and on the multiplication algorithm of whole numbers.

- To reflect and to discuss about the characteristics of the deductive reasoning to prove a mathematical statement compared to an empirical verification based on cases.
- To promote the algebraic reasoning by introducing symbolic notation that facilitates the generalization of an arithmetic problem.
- To analyze the "mathematics in use" put into effect in the solution of a problem, that is to say, the prior or required mathematic objects and processes needed to get involved in the solution, as well as the new objects and processes that emerge during the solution process.

First the students worked in groups of two to four members, and they had to write the solution in every detail. Second, they were given the following question: "what mathematics have you used in the solution of the problem?" The purpose of this question is for the students to recognize, besides of the concepts and procedures involved, the role of the various languages used and the meanings ascribe to terms and expressions, the types of justification of the properties and procedures used, the argumentation and generalization[2] processes.

After thirty minutes, the students handle in the answer sheets, which were used by the teacher to organize the group discussion and the institutionalization.

### 2.1. An introductory situation to multiplication

As an introductory situation to the multiplication of natural numbers, we proposed, to a group of 84 students, the following problem (Malaspina, 2007):

Peter write on the blackboard the numbers 2, 5, 6 and 3.

a) Choose three different numbers, write them in the following pigeonhole, so that the resulting product be the **greatest as possible**

$$\begin{array}{r} \square\square\ \times \\ \square\phantom{\ \times} \\ \hline \end{array}$$

b) John says he is able to choose three numbers among the given, so that its product be the greatest, so that there is no need to carry out any multiplication.

b1) Is it possible? b2) which could be the John's procedure? ; b3) which could be the justification? Give an extended (ample) answer.

c) Peter proposes the following challenge:

Given five natural numbers of one digit each (***a, b, c, d, e***), three of them must be chosen to write the **multiplicand** and another one to write the **multiplier**. Describe and justify a procedure to choose the numbers so that the resulting product **is the minimum**.

### 2.2. Expected solutions

The first "naive" solution that we found, as is described in Malaspina (2007), is to try out different configurations of groups of three numbers, to calculate the products and to find the biggest among

[2] The mathematic objects and processes upon which are based the reflection are described in Godino, Batanero and Font (2007), as well as the anthropological assumptions that support the "onto-semiotic approach". The students are progressively introduced into recognizing such objects and processes, as well into the plural and relativistic perspective on the meaning of mathematics objects.

them. If the 24 configurations are tried out, this procedure of exhaustive case by case verification gives the desired solution. But it is seen that the procedure is very much inefficient.

For the question in section in b) the John's procedure will be:

- Rule out the number 2, being the lowest it will render the lowest product.
- Choose as multiplier the biggest number, 6, in the tens of the multiplier the five, and in the units place, the 3. 6 and 5 cannot be commutating because 6x3 is bigger than 5x3. The solution is 53x6=318.

For the generalization that is being asked in section c) the procedure will be the following:

1) Order the given numbers from lowest to biggest; for instance, the alphabetic order of the letters can be matched to the order of the numbers, that is to say: $a < b < c < d < e$

2) Rule out the two biggest numbers, *d* y *e*, due to the property "the lower the factor, the lower the product", and in this case we want to find the lowest possible product.

3) Choose the multiplier as the lowest number, *a*; as the tens of the multiplier *b* and as unit, *c*. The solution is *bc* x *a*.

In the following section we carry out an a priori analysis of the mathematic objects and meanings that are put into effect in the activity; this analysis will be then used to interpret the students' answers and to organize the corresponding institutionalization process.

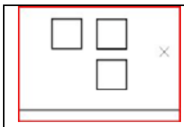
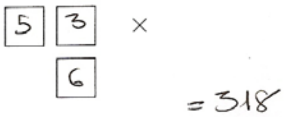
## 3. MATHEMATICS OBJECTS AND MEANINGS

The notion of configuration of objects and processes as a mean to describe the mathematical practices for the solution of a problem has been introduced in the "onto-semiotic approach" of mathematical knowledge.

This notion allows broadening the focus of attention from the representations towards the conglomerate of entities referred to by the entities themselves and the roles played in the mathematic activity (Font, Godino and D'Amore, 2007). Then, we identified the types of objects and meanings involved in the solution of a problem, clustered in the following types: languages, concepts, procedures, properties and arguments; we distinguished as well, the entities that can be considered as prior or intervening and the new entities, emerging from the activities.

### 3.1. Linguistic terms

| OBJECTS: | MEANINGS: |
|---|---|
| *Previous:* | |
| "Choose three out of the numbers…" | Choosing a sample made of three digits out of five, use two of them as multiplicand and the other as multiplier. |
| product of numbers; | Result of the operation "multiply". |
| biggest possible | Greatest of the set of products obtained by arranging all the possible selection made of three numbers. |

| | |
|---|---|
|  | Indicate the variables of the different digits that should comprise the multiplicand, the multiplier, the operation (x) and the place where the product must be written. |
| alphabetical symbols (a, b, c, d, e ) | “ any natural numbers”. |
| *Emergent:* | |
|  | Selection of numbers that yields the biggest number. |
| Systematic writing of every possible selection, or one expression in natural language based on the property “ the bigger the factor, the bigger the product” | Argument that states that the product of the chosen selection is the biggest. |

*Likely conflicts:*

- It can be expected that the pupils partially write the set of combinatorial selections without providing reasons that justifies that it is not necessary to write down all of them.
- The term “any numbers (a, b, c, d, e)” can be interpreted in the sense that some particular values could be given (1, 2, 3, 4, 5), “any numbers you want” and to show the solution for that case.
- Partially or incorrect written explanation on the procedures and justifications, in both cases b) and c).

### 3.2. Concepts/ definitions

| OBJECTS: | MEANINGS: |
|---|---|
| *Previous:* | |
| Numbers | Recursive sequence of symbols that are combined according to certain rules to form another one made of two and three ciphers; decimal numeration, units, tenths and hundredths. |
| Natural numbers multiplication, factors, product | Arithmetic operation; given two numbers (multiplicand, multiplier) the multiplication of such numbers produces a third number (the product)… |
| Equality | Result of an arithmetic operation. |
| Variable | Symbol (literal) that can take values out of a set of numbers. |
| Function;<br>P = f($x$, $y$, $z$); defined on A = {2, 5, 6, 3}, in the interval [0, 9] in $N$. | The product depends on the values assigned to the units and tenths places of the factors. |
| Ordering, biggest, maximum of a set | Ordering and set’s biggest upper value, made of by products of every possible selection. |
| *Emergent:* | |
| Combinatory selection of sets of three numbers | To propose the multiplicand and multiplier of the multiplications. |
| Set of selections | Domain of definition of the function P whose maximum is to be found. |
| Multiplicand and multiplier | In this case they fulfill a necessary role (it is not the same to write 6 in the multiplier as unit that to write it in the tenths place of the multiplicand) |
| Procedure<br>Justification | Using these concepts two new problems are proposed: find the most effective procedure that writing every possible selection and justify the new procedure validity inductively instead of empirically. |
| Reasoning | Detailed description of both, procedure and justification. |

*Likely conflicts:*

- Students may not be familiarized with the meta-mathematic concepts, procedure, justification, reasoning.

- The function concept is used in a tacit way (not ostensive); as it is a three variables function P(x, y, z) the students could not identified this function due to the fact they are not acquainted with this kind of functions.

### 3.3. Properties

| OBJECTS: | MEANINGS: |
|---|---|
| *Previous:* | |
| Decimal system rules, (one tenth = 10 units) | They are used in writing the positional numeric system and the multiplication algorithm. |
| Basic arithmetic facts (multiplication tables and adding tables) | They are used in the multiplication algorithm. |
| Associative, commutative and distributive properties | Justify the multiplication algorithm. |
| *Emergent:* | |
| P1 "Bigger (lower) factor, bigger(lower) product" | Justify the procedure that provides the best solution. |
| P2 : The choice of numbers that have to be taken is 53 x 6 because its product is the biggest. | This statement represents the solution to the problem 2). |
| P3: If a<b<c<d<e, *bc* x *a* yields the lowest product. | This statement represents the problem solution. |

*Likely conflicts:*

- Do not find the properties P1, P2, or P3.

### 3.4. Procedures

| OBJECTS: | MEANINGS: |
|---|---|
| *Previous:* | |
| Multiplication algorithm of two cipher number by one cipher number | It is used to obtain the products asked for. |
| Systematic writing of every possible choice of three given numbers (permutations of 4 numbers taken in groups of two, e.g., 24) | It is used to write every possible product and to find the maximum. |
| *Emergent:* | |
| b2) Discard number 2; takes 6, the bigger, as multiplier; take 53 as multiplicand | It provides the best problem solution (53 x 6 = 318). |
| c) Ordering; *a<b<c<d<e*, discard *e*, … | It provides the general solution in section c). |

*Likely conflicts:*
- Be unable to figure out every possible combination.
- Do not find the procedure for case b2).
- Do not find the procedure for case c).

### 3.5. Arguments

| OBJECTS: | MEANINGS: |
|---|---|
| *Previous:* | |
| Empiric exhaustive verification of, either, every 24 possible factor choice, or some of them. | It provides an ineffective problem solution to question a). |

| | |
|---|---|
| *Emergent:* | |
| A1: Based on the property, “bigger the factor, bigger the product” the number 2 is discarded, taking as multiplier number 6 and writing 5 in the tenths place of the multiplicand. | Deductive justification of the proposition that states the best problem solution in case b3). |
| A2 If $a<b<c<d<e$, $e$ is discarded as it is the biggest will render the biggest product. | Deductive justification of the proposition that states the best problem solution in case c). |

*Likely conflicts:*

- Do not properly state the justifications A1 and A2.

4. ANALYSIS OF THE STUDENTS’ ANSWERS

The problem analyzed in the previous sections was used as an introductory situation in one of the topics of mathematics for teachers in a training course for in-training teachers.

During the first stage, the students solved the problem working in teams of 2 to 4 students; they should, at the end of this first stage, handle in the solution to the teacher, who organized the debate taking into account some of the solutions given by the teams. This didactical technique allows gaining access to the set of initial students´ meanings on the different objects put into effect during the solution and based on it, to promote a new learning.

In what follows we summarize the students´ answers given by the whole group of students.

For section a) of the problem, we found that 5 out of 24 teams (20%) performed just a multiplication, realizing that it is possible to obtain the configuration of numbers that gives the biggest product keeping in mind the properties of the numeration system and of the multiplication of natural numbers. However nine teams (37.5%) tried out more than four times (some, even up to 9), and ten (41, 6%) tried out 2 or 3 multiplications.

In table 1 we summarize the types of answers, the frequency and percentages of each type, in sections b) and c). We would like mentioning that 33,2% of the students do not justify the procedure that gives the biggest product without carrying out any multiplication, or justifies it incorrectly , and the 41, 6% could not obtain the generalization.

Table 1: Types of answers, frequency and percentage (sections b) and c))

| Section b) | Frequency | % |
|---|---|---|
| Correct justification | 12 | 50,0 |
| Partial ability to give explanations | 4 | 16,6 |
| Do not justifies or justifies incorrectly | 8 | 33,2 |
| TOTAL | 24 teams | |

| Section c) | Frequency | % |
|---|---|---|
| Correct generalization | 4 | 16,6 |
| Partial ability to generalize | 10 | 41,6 |
| Do not generalizes | 10 | 41,6 |
| TOTAL | 24 teams | |

Table 2 includes the answers given by one of the teams that could not obtain either the general answer to the problem or elaborate appropriate arguments (justifications). This group carried out eight multiplications before reaching the solution 53 x 6 =318, and did not discard 2 as a possible digit in the multiplicand, in the units place as well as in the hundredths place. The students did not argument to justify why they did not keep on trying the remaining cases. In section b) they described, in incomplete way, the procedure (write the biggest number…). In section c) the expression "given any five natural numbers", even though the use of letters are suggested to express such numbers, the students chose to reason over a particular example, the numbers 1, 2, 3, 4, 5 and again, in an incomplete way.

Table 2: Answers of a team that did not obtain the generalization

| | |
|---|---|
| a) $\boxed{5}\boxed{3} \times \boxed{6} = 318$ | 65 × 3 = 195; 56 × 3 = 168; 63 × 5 = 315; 36 × 5 = 180; 53 × 6 = $\boxed{318}$; 52 × 6 = 312; 25 × 6 = 150; 25 × 3 = 75 |
| b) b1) ¿Es esto posible? Sí,<br>b2¿Cuál podría ser el procedimiento de Juan? Poner el número más grande de manera que cuando multiplique por la decena, saldrá el número mayor.<br>b3)¿Cómo se puede justificar? | |
| c) Los números serían: 1, 2, 3, 4, 5<br>$\boxed{2}\boxed{3}\boxed{4} \times \boxed{1} = 234$<br>Colocaría el menor de los números en el multiplicador, y el siguiente más pequeño en el multiplicando (decena), así, al multiplicar saldrá el menor número. | |

In Table 3 we summarize the types of objects, meanings and conflicts that we found in the answers given by the team, as an illustrative example of the description of cognitive configurations, obtained by using the tools described in section 4.

Table 3: Cognitive configuration of a team that did not obtain the generalization

| **Types of objects** | **Meanings/conflicts** |
|---|---|
| LINGUISTIC:<br>- Examples of possible configurations of numbers used as multiplicand and multiplier.<br>- Description of incomplete procedures and justifications.<br>- Factors organized in columns | - Attribution of the correct meaning to the terms and expressions used en the statement, including the writing in columns´ format that students reproduced closely.<br>- Inflexible in their way to write multiplications, following the problem statement format.<br>- Incomplete and deficient ways of expressing procedures and justifications. |
| CONCEPTS:<br>- Tenths, hundredths<br>- Multiplication, biggest product<br>- Procedure<br>- Set of possible configurations; "any number" | - Do not recognize the complete set of possible configurations.<br><br>- The generic number is interpreted as a "specific case". |

| | |
|---|---|
| PROCEDURES:<br>- One digit multiplication | - Do not describe the general procedure to obtain the lowest product with the given conditions (assuming an ordering in the variables *a*, *b*,… etc). |
| PROPERTIES<br>- Decimal number numeration rules and multiplication algorithm. | - Do not recognize the property, “bigger factors, bigger product” that permits to discard the number 2. |
| ARGUMENTS:<br>- Partial verification of cases | - They are unable to elaborate the required argument in a complete way. |

## 5. EPISTEMIC AND COGNITIVE REFLECTION IN TEACHERS’ TRAINING

During the first stage of the formative cycle, which we implement in training teachers, we propose them a mathematical problem in order to develop mathematical competence in topics related to its career as teachers. The didactic trajectory of the learning process implemented for the given problem situations considers the following phases: problem exploration, team work, formulation and validation. These moments of socio-constructivist type are complemented with moments of institutionalization, practicing, and individual study of properly selected texts that give an instructional component to our didactical model.

However the training of a mathematics teacher should not be limited to developing mathematical competences, acquired by means of a specific didactic model. In addition, it is necessary that student teachers develop competences of analysis and reflection about the mathematical activity itself and about the mathematical knowledge put into effect while solving problems. Using these competences they can select or adapt mathematical problems to their students’ needs, and reconstruct the objects and meanings involved in problem solving.

To this purpose we have designed a type of epistemic-cognitive analysis situation, which is given to the student teachers after solving a mathematical problem, and based on the following question:

> *What mathematic knowledge is put into effect in the problem’s solution? Complete the following table naming the mathematic objects and meaning involved?*

| Objects(Previous and emergent) | Meanings: |
|---|---|
| *Situations-problems* | |
| *Linguistic elements* | |
| *Concepts-definition* | |
| *Properties* | |
| *Procedures* | |
| *Arguments* | |
| Conflicts: | |

This question is intended for the students to recognize, besides the concepts and procedures, the various language's roles and meanings attributed to terms and expressions, the types of properties and procedures justifications, the processes of proving and generalization. It is all about to create a departing situation to progressively enable the student teacher to carry out the type of analysis illustrated in section 4. The purpose is to give the teacher the opportunity to recognize the complex web of objects and meanings that is put into effect in the study processes they are supposed to design, implement and assess.

To tackle this epistemic-cognitive analysis we implement, once more, a didactic trajectory that takes into account the following steps:

- Individual exploration.
- Team work to discuss the proposals and to elaborate a shared answer.
- Presentation and discussion in the class.
- To institutionalize the knowledge by the teachers' educator.

This problem-situation of "epistemic analysis" is been experimented with a number of teacher students' groups and with different elementary mathematic problems. Among the first conclusions we can mention: the activity is a challenge for the student teachers; the identification and discrimination of the different types of objects and meanings are difficult tasks for the student teachers to perform. The identification and discrimination of mathematical objects and meanings are controversial, because it requires certain level of meta-cognitive activity that they are not familiar with.

## 5. SOME IMPLICATIONS TO TRAINING STUDENT TEACHERS

The analysis that we have included in the previous sections, using some theoretic notions of the "onto-semiotic approach", has been done by the researches as an element of reference and reflection on the types of mathematic objects and meanings used by the students. We have come to determine the lack of mathematic knowledge, in particular the difficulties that the students exhibit to use the symbolic notations as well as the resources for the generalization; we have also identified the anchoring of students' thinking in reasoning of empirical type.

Even though this type of analysis reveals its usefulness for the teacher educator, we consider that it is possible and desirable to train future teachers to carry out such analysis on their own teaching and learning experiences. In our ongoing research project about "Assessment and development of competences of didactic analysis for the mathematics teacher" the problem solving has a central role to help developing mathematic competences. But the problem solving activity is complemented with the epistemic-cognitive reflection prompted by the questions: What mathematics is involved in the solution process? What mathematics is used by the pupil?; these questions are supported by the use of the tools provided by the "onto-semiotic approach" to mathematical knowledge (Godino, Batanero and Font, 2007).

The kind of analysis proposed of the “mathematics in action” that we carry out on this report should be a teacher’s instrumental competence by allowing him to recognize the complexity of the mathematic objects and meanings put at stake during the problem solving process, foresee conflicts, adapt them, not only to the student’s skills but also to the teaching objectives. The design and implementing of didactic situations for the student-teachers training is the core of the activity, whose central role is the meta-analysis (Jaworski, 2005) of a key component for the teaching: the mathematic activity understood not only from the institutional point of view (socio-epistemic) but also from the personal perspective (or cognitive).

Godino, Bencomo, Font y Wilhelmi (2006), claim that it is necessary for the teacher to plan the teaching having in mind the institutional meanings that are intended to be studied, adopting a wide view, not one limited to the discursive aspects (epistemic suitability). Furthermore, it is necessary to design and to implant a didactic trajectory that includes the prior students´ knowledge (cognitive suitability), to identify and to resolve the semiotic conflicts that sprout along the whole process, using the material and temporal resources necessary (interactional and meditational suitability). But the assessment of the appropriate epistemic and cognitive aspects of a study process demands to carry out some previous analysis such as: the type of mathematic problems, the operative and discursive practices implemented, as well as the identification of the network of objects and meanings put into effect.

The analysis of the problems, which is the focus of the corresponding didactics configurations, is a previous and necessary step in the elaboration of tools intended to assess the learning. The epistemic and cognitive model that characterizes the “onto-semiotic approach” of mathematic knowledge provides analysis tools, as those applied on the example discussed above, that allow the design of didactic trajectories and assessment tools that are ecologically suited, that is to say, adapted to the context, to the prior mathematic competences of the students and to the intended or implemented learning objectives.

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