LEVELS OF ALGEBRAIC REASONING IN PRIMARY AND SECONDARY EDUCATION[1]

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Based on the onto-semiotic approach to mathematical knowledge and instruction a characterization of algebraic reasoning in primary education has been proposed, distinguishing three levels of algebraization. These levels are defined taking into account the types of representations used, generalization processes involved and the analytical calculation at stakes in mathematical activity. In this paper we extend this previous model by including three more advanced levels of algebraic reasoning that allow to analyze mathematical activity carried out in secondary education. These new levels are based on the consideration of 1) using and processing parameters to represent families of equations and functions; 2) the study of algebraic structures themselves, their definitions and properties.

Key words: algebraic reasoning, primary education, secondary education, onto-semiotic approach, teachers' education

INTRODUCTION

Recognizing characteristic features of algebraic thinking is an issue that has attracted attention to many researchers in the field of mathematics education, because it is necessary to promote such reasoning at different levels of elementary and secondary education (Kieran, 2007; Filloy, Rojano & Puig, 2008). Depending on how school algebra is conceived, decisions concerning how to introduce such algebra will be taken since early levels, or be delayed until secondary education; it may also change the corresponding instructional strategies. In fact, the research and development program based on "early algebra" (Carraher & Schliemann, 2007; Cai & Knuth, 2011) is supported by a conception of algebra recognizing signs of algebraic thinking in mathematical activities on initial educational levels, as shown in NCTM (2000). However, while progress has been made in the characterization of school algebra, the problem is not completely solved, particularly because algebras in primary and secondary education are related.

In previous publications (Ake, Godino, Gonzato & Wilhelmi, 2013; Godino, Ake, Gonzato & Wilhelmi, 2014) a model of algebraic thinking for primary education has

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been proposed, with three distinguished levels of algebraic thinking. Furthermore, criteria was stablished to delimit these algebraic levels from 0 (arithmetic nature of mathematical activity) to 3 (clear algebraic activity), with two intermediate levels of proto-algebraic activity. The criteria to define these levels are based on the type of mathematical objects and processes involved in mathematical activity, according to the onto-semiotic approach (OSA) of mathematical knowledge (Godino, Batanero and Font, 2007). Algebraization levels are assigned to operative and discursive practices performed by a mathematical subject (individual or epistemic) that solves a mathematical task, rather than the task itself, which can be solved in different ways, and may bring into play different algebraic activity.

In this paper, we extend that model of algebraization levels for mathematical activity to secondary school, also supported by the onto-semiotic distinctions suggested by the OSA, particularly in the presence, use and processing of parameters in functions and equations. The work is organized in four sections. The following section summarizes features of algebraic reasoning levels in elementary education; next, we define the three new levels of algebraization, include some illustrative examples and connect these levels to the presence of discontinuities in the onto-semiotic configurations involved in mathematical practices.

LEVELS OF ALGEBRAIC REASONING IN PRIMARY EDUCATION

Table 1 summarizes the essential features of the three preliminary algebraization levels described by Godino et al. (2014), completed by level 0 (absence of algebraic characteristics). An example is also included to help to understand the distinction among levels. In summary the definition of levels are based on the following onto-semiotic distinctions:

- Presence of intensive algebraic objects (i.e., entities of general or indeterminate character).
- Transformations (operations) applied to these objects based on the application of structural properties.
- Type of used language.

Table 1: Characteristic features of elementary algebraic reasoning levels

Task: In a certain school, students either go by car or by walk. For every student who goes by car there are 3 going by walk. If there are 212 students at that school, how many of them use each means of transportation?			
LEVELS	OBJECTS	TRANSFORMATIONS	LANGUAGES
0	No intensive objects are involved. In structural tasks unknown data can be used.	Operations are carried out with extensive objects.	Natural, numerical, iconic, gestural; symbols referring extensive objects or unknown data can take part.
	Example of resolution:		

<table>
<tr><td></td><td colspan="3">For every 3 students who walk, there is 1 that goes by car. Hence, in every bunch of 4 students (3 + 1) there is 1 which goes by car (a fourth part). Thus, 50 out of 200 students go by car and 3 out of 12 students would also go by car. Therefore, 53 students would go by car and three times that amount, that is, 159, by walk.</td></tr>
<tr><td rowspan="2">1</td><td>In structural tasks unknown data can be used.
In functional tasks intensive objects are recognized.</td><td>In structural tasks relations and properties of operations are applied.
In functional tasks calculation involve extensive objects.</td><td>Natural, numerical, iconic, gestural; symbols referring to intensive recognized that can be used.</td></tr>
<tr><td colspan="3">Example of resolution:
For every 4 students there are 3 which go by walk. We write out the following proportion:
4 (children) ------> 3 go by walk
212 (children) ------> x go by walk
$\frac{4}{3} = \frac{212}{x}$; x = 3 × 212/4 = 159.
Once we obtain the number of children who go by walk, the number of students going by car is easily obtained, 212 - 159 = 53.</td></tr>
<tr><td rowspan="2">2</td><td>Indeterminate or variables are involved.</td><td>In structural tasks equations are of the form $Ax \pm B = C$.
In functional tasks generality is recognized but operations with variables are not carried out to get canonical forms of expressions.</td><td>Symbolic - literal, used to refer the intensive recognized, although linked to the spatial, temporal and contextual information.</td></tr>
<tr><td colspan="3">Example of resolution:
$212 = x + 3x$
$212 = 4x$; $x = 212 / 4$; $x = 53$
53 children go by car and 212-53 = 159 by walk.</td></tr>
<tr><td rowspan="2">3</td><td>Indeterminate or variables intervene.</td><td>In structural tasks equations are of the form $Ax \pm B = Cx \pm D$.
Operations with indeterminate or variables are carried out.</td><td>Symbolic – literal; symbols are used analytically, without referring to contextual information.</td></tr>
<tr><td colspan="3">Resolution example:
x = Children going by car
y = Children going by walk
$x+y = 212$ $x+3x = 212;$
$y = 3x$ $4x = 212; x = 212/4 = 53$</td></tr>
</table>

The algebraization levels we propose are related to two aspects that Kaput (2008) identifies as characteristic of algebraic reasoning, namely, algebra as:

a) Systematic symbolization of generalizations of regularities and constraints.
b) Syntactically guided reasoning and actions on generalizations expressed in conventional symbolic systems.

Aspect a) is specified in our model as levels 1 and 2 of proto-algebraic reasoning, while b) is associated with level 3, where algebra is already consolidated.

Obviously, these levels do not exhaust the processes of "algebraization". Instead, they do refer the gradual enrichment of tools for solving problems with an increasing degree of symbolization in other contexts of use. These processes, taken from the end of primary school and junior secondary school, must evolve to higher levels of algebraization.

LEVELS OF ALGEBRAIC REASONING IN SECONDARY EDUCATION

In this section, we extend the model of algebraization levels to secondary and high school mathematical activity, recognizing three additional algebraization levels for this educational stage.

Using parameters and its treatment can be a criterion for defining higher levels of algebraization as it is linked to the presence of families of equations and functions, and therefore imply new "layers" or levels of generality (Radford, 2011). The intervention of parameters will be linked to the fourth and fifth level of algebraization, while the study of specific algebraic structures leads to recognizing a sixth algebraization level of mathematical activity.

Fourth level of algebraization: using parameters

The use of parameter for expressing equations and function families is indicative of a higher level of algebraic reasoning, regarding the third algebraization level considered by Ake et al. (2013), which is linked to the processes of "operate with a unknown or variable." This is a “first encounter” with parameters and coefficient variables involving discrimination of domain and range of the parametric function, i.e. the function that assigns to each value of the parameter a specific function or equation. As Ely and Adams (2012, p. 22) claim, *“A significant conceptual shift must occur in order for students to be comfortable using placeholders in algebraic expressions rather than just numbers”.*

Example 1: The linear function

In the algebraic expression, $y = 2x$, the literal symbols x and y are interpreted as variables, as symbols that can take any value from the previously established number set, usually R. The numerical values x and y co-vary one in terms of the other, according to the rule laid down in the corresponding expression, in this case, multiplying by 2 the value assigned to x. The x coefficient can be generalized to any value within a certain domain, which is indicated by an expression of the type $y = ax$; letter a intervenes as a parameter: it can take different values within a certain domain, so that for each taken value a particular function is obtained. For example, for $a = 2$ we have $y = 2x$.

Thus, we could say that a parameter is a literal symbol involved in an expression with other variables, such that for each particular value assigned to it, a function is obtained. It is therefore a mean of expression of a functions family F = {f(x) =

$ax/a \in R\}$, or more precisely, a family of functions family depending on the domain D of definition of the functions f: $F_D = \{f(x) = ax \mid a \in R; x \in D\}$.

The letter symbols *x* and y (*f(x)*) are indicative variables of a first level of generality, whose domains of definition and range are the numeric sets in which they are defined. The symbol *a* is also a variable, but with a second level of generality, whose domain of definition could either be the same as before (D) or just another number set, and the range of values is the family of functions F_D.

Example 2: Quadratic equation

Parameters are used not only to express and operate with function families, but also equation families (Ely and Adams, 2012). For example, $ax^2 + bx + c = 0$ ($a \neq 0$) is the general expression for the quadratic equations family. There is only one unknown, *x*. Letters *a*, *b*, *c*, usually considered as varying coefficients, take specific values within a set of possible values (real numbers and $a \neq 0$) to produce a particular equation.

It is said, therefore, that a parameter is a variable that is used with two or more other variables to specify a family of functions or equations. For families of equations commonly the parameter is named coefficient. In a way, the parameter plays the role of independent variable in a function whose domain is the set in which the parameter takes values and their rank is a set of functions. For each value assigned to the parameter a function image is obtained. Therefore, the expression $y = ax^2+bx+c$, is not a function but a family of functions, though it is usually referred to as "the quadratic function." It is an expression in which three parameters indicated by the letters *a*, *b*, *c* are involved. Giving a particular value to each of the parameters a specific quadratic function is obtained.

Example 3: General matrix with n rows and m columns

Matrixes are more than simple arithmetic objects, since the relative position of a particular number in the box gives structured information. Furthermore, the study of matrixes in secondary school is full of symbolic notation, because each element in a cell on a general matrix has its own index, $A = (a_{ij})_{m,n}$, indicating row and column. These indexes are parameters and, therefore, the discursive practice defining a matrix belongs to a fourth algebraization level, even though the study of operations and structural properties of matrixes could belong to a higher level, as seen in Figure 1.

Matrixes are numerical rectangular tables: A=(a11 a12 a13…a1n; a21 a22 a23…a2n; a31 a32 a33…a3n; …; am1 am2 am3…amn). This matrix has m rows and n columns. It has dimension mxn. Elements aij are real numbers (aijeR). The matrix below could be simplified as (aij)i=1,…m;j=1,…,n, A(aij)m,n, (aij). When m=n, the matrix is named square.	Two matrixes are identical when their dimensions are equal and their elements agree one by one. A=(aij)m,n, B=(bij)m,n; A=B iff aij=bij The transpose of matrix A=(aij)m,n is a matrix At=(aji)n,m where rows and columns have been interchanged. A matrix A is symmetric if At=A. A symmetric matrix must be square.

Figure 1. Matrix (Colera and Oliveira, 2009, p.50)

This example suggests the possibility to analyze algebraization levels when working with matrixes and when applying them in the resolution of equation systems.

Fifth level of algebraization: treatment parameters

We can link a higher level of algebraization to mathematical activity displayed when analytical (syntactic) calculations are carried out in which one or more parameters are involved. Operations with parameter involve greater semiotic complexity level, since objects emerging from these systems of practices put at stake algebraic objects of the previous level (equations or functions families).

Example 3: Obtaining the general formula for quadratic equations

We proceed by symbolic manipulation and successive equivalences. Assuming the director coefficient *a* is not 0 ($a \neq 0$) — otherwise the equation would not be quadratic — we have:

$$ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a} \Leftrightarrow$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \Leftrightarrow x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \Leftrightarrow$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Leftrightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, in this case the structure of the solutions is written in terms of the coefficients linked by rational operations (addition, subtraction, multiplication, division) and square roots calculation.

Example 4: Geometric progressions

Defining the general term of a geometric progression (figure 2) is performed by discursive practices in which two parameters, a_1 (first term of the sequence) and r (progression ratio) are involved. The sequence is a function whose domain is *N* and range is *R*; therefore the parameters a_1 and r define a family of functions (sequences), so this discursive practice put at stake algebraization level 4. The statement and proof of the property that states the sum of the first *n* terms of a geometric progression ($r \neq 0$) involves calculating with parameters, as shown in figure 1, so implying the level 5 of algebraization.

36.8 Finite Geometric Series When we sum a known number of terms in a geometric sequence, we get a finite geometric series. We know that we can write out each term of a geometric sequence in the general form: $an = a_1 \cdot r^{n-1}$ where • n is the index of the sequence; • a_n is the nth-term of the sequence;	By simply adding together the first n terms, we are actually writing out the series $Sn = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$ We may multiply the above equation by r on both sides, giving us $rS_n = a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1} + a_1r^n$ .... Dividing by (r − 1) on both sides, we arrive at the general form of a geometric series:

• a_1 is the first term; • r is the common ratio (the ratio of any term to the previous term).	$S_n = \sum_{i=1}^{n} a_1 \cdot r^{i-1} = \frac{a_1 (r^n - 1)}{r - 1}$

Figure 2: Finite geometric series (Free High School Science Texts, Mathematics Grades 10 – 12, p. 469, 2008)

Sixth level of algebraization

The introduction of certain algebraic structures (such as vector spaces, or groups) and the study of function algebra (addition, subtraction, division, multiplication, and composition) start at high school, bringing into play algebraic objects and processes of higher level of onto-semiotic complexity than those considered at level five. It may be useful, therefore, to characterize a sixth algebraization level that should help us to focus our attention on the specific nature of the mathematical activity involved. High school books include texts and activities corresponding to this sixth algebraization level:

Example 5: Vector space

Figure 3 shows a general formulation of the algebraic structure of a vector space. This is a first encounter with the algebraic structure of vector spaces which brings into play a set of mathematical objects (vectors) on which operations satisfying a set of specific properties are defined. This requires an initial "structural study" of vectors, since in this type of (axiomatic) presentation properties of vector addition and multiplication of vector by numbers have to be established.

The vector idea as an arrow leads to vector space: sets of vectors among some operations satisfying certain properties are defined. But there are other mathematic entities with the same operations and properties. So, the vector space definition is much broader and open than collections of “arrows”. We have a set, V, among whose elements (called vectors) there are two operations defined:

SUM OF TWO ELEMENTS OF V: if $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$

PRODUCT BY A REAL NUMBER: if *a*ϵR and *u*ϵV, then *a.u*ϵV

If (V, +, .) satisfies the following properties then is a vector space on R.

SUM OF VECTORS	
ASSOCIATIVE	$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
COMMUTATIVE	$\vec{u} + \vec{v} = \vec{v} + \vec{u}$
NULL VECTOR	It is a vector called $\vec{0}$ such that if $\vec{v} \epsilon V$ fulfils: $\vec{v} + \vec{0} = \vec{v}$
OPOSITE ECTOR	All $\vec{v}$ has its opposite $\vec{-v}$: $\vec{v} + \vec{-v} = \vec{0}$
MULTIPLYING A VECTOR BY A NUMBER	
ASSOCIATIVE	$(a.b).\vec{v} = a.(b.\vec{v})$
DISTRIBUTIVE I	$(a+b).\vec{v} = a.\vec{v} + b.\vec{v}$
DISTRIBUTIVE II	$a.\left(\vec{u} + \vec{v}\right) = a.\vec{u} + a.\vec{v}$
PRODUCT BY 1	If $\vec{v} \epsilon V$ then $1.\vec{v} = \vec{v}$

Figure 3: Vector space (Colera & Oliveira, 2009, p. 62)

Example 6: Composition of functions

In Figure 4 the notion of function is put into play in all its generality, replacing a particular family of functions by any function. Operations are carried out over functions to produce new functions whose properties will be studied in general. For example, statements such as “the composition of functions is not commutative” would arise. In fact, a set of functions (polynomials, for example) satisfying certain operations (addition, multiplication, etc.) is an "algebra".

It is possible to combine two functions by adding, subtracting, multiplying or dividing two given functions.
There is another way to combine two functions to create a new function. It is called composition of two functions. It is a process through which we will substitute an entire function into another function.
First let’s get acquainted with the notation that is used for composition of functions. When we want to find the composition of two functions we use the notation $(f \circ g)(x)$.
Another way to write this is $f(g(x))$. This is probably the more practical notation although the first notation is what appears most often in books.

Figure 4: Composition of functions (AlgebraLAB. Project Manager. Mainland High School)

Algebraization levels and onto-semiotic discontinuities

Algebraization levels are basically levels of generality, combined with the use of various registers of semiotic representation (RSR), theirs transformations and conversions (Duval, 1995). Under the OSA these levels can be characterized by the presence of different types of onto-semiotic configurations (Godino, Font, Wilhelmi and Lurduy, 2011) which involve practices, objects and processes implying new levels of generality or syntactic calculus, supported by symbolic representations of the corresponding objects. This furthermore implies the intervention of unitization, materialization and reification processes involved in the generalization and representation processes (Godino et al., 2014).

The proposal of considering algebraization levels of mathematical activity can help raise awareness of gaps or discontinuities within the sequence of configurations on epistemic trajectories of the corresponding processes of mathematical study (Godino, Contreras and Font, 2006). These gaps refer to the use of different registers of semiotic representation, their treatment and conversion, as well as the intervention and establishment of relations between conceptual, propositional, procedural and argumentative objects of greater generality (intensive objects emerging from other intensive). In other words, these gaps can be explained by analyzing how, numerical-iconic and analytical - algebraic onto-semiotic configurations involved are articulated, and not only for the treatment or conversion of RSR.

Recognizing algebraization levels can be helpful to analyze the articulation of these onto-semiotic configurations. The identification of objects, processes and meanings involving access to different levels of algebraization allows the design of operative, normative and discursive practices aimed at learning progression. This progression will involve coping with discontinuities in levels of generality, representation, calculation and construction of algebraic objects in different educational institutions.

SYNTHESIS AND IMPLICATIONS FOR TEACHER EDUCATION

In this work we have completed the work started in Ake et al. (2013) and Godino et al. (2014) on the identification of algebraization levels of mathematical activity in primary education, including three new levels that characterize secondary mathematics. As a summary we propose the following six levels of algebraic thinking in primary and secondary education (along with level 0, indicating absence of algebraization):

Level 0: Operations with particular objects using natural, numerical, iconic, gestural languages are carried out.

Level 1: First encounter with the "generic number", the algebraic structure properties of *N* and the algebraic equality (equivalence). That is, relational thinking.

Level 2: First encounter with the alphanumeric representation of functions and equations and simplifying expressions.

Level 3: First encounter with the treatment of unknowns and variables using structural properties (cancellation, replacement, etc.) and the algebraic and functional modeling.

Level 4: First encounter with the use of parameters in functions and variable coefficients, that is, with the expression of families of equations and functions, which are second order intensive objects.

Level 5: First encounter with the joint treatment of unknowns, variables and parameters as well as the structure of the solution emerging from the parameter treatment.

Level 6: First encounter with the study of algebraic structures themselves, their definitions and structural properties.

These algebraic reasoning levels have implications for teacher training, both in primary and secondary education. It is not enough to develop curriculum proposals (NCTM, 2000) including algebra from the earliest levels of education; the teacher is required to act as the main agent of change in the introduction and development of algebraic reasoning in elementary classrooms, and its progression in secondary education. Reflexing on the recognition of objects and processes of algebraic thinking can help identify features of mathematical practices on which teachers can intervene to gradually increase the algebraization levels of students' mathematical activity.

Recognizing algebraization levels 4, 5 and 6 by secondary school teachers, along with its articulation with the previous levels, can help raise awareness of the gaps or onto-semiotic discontinuities which may occur when carrying out tasks proposed to students.

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