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TABLE OF CONTENTS
VOLUME 2

RESEARCH REPORTS

A - HAN

PROTO-ALGEBRAIC LEVELS OF MATHEMATICAL THINKING

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Researches on the nature and development of algebraic reasoning in early grades of primary education have been inconclusive about the boundaries between mathematical practices of algebraic nature and those not algebraic. In this report we define primary levels of algebraization in school mathematics activity and prototypical examples of answers to a task for each level, based on the type of objects and processes proposed by the onto-semiotic approach of mathematical knowledge. This model can be useful to develop the meaning of algebra in elementary school teachers and empower them to promote algebraic thinking in primary education.

Key words: elementary algebra, mathematical practice, reasoning level, teacher's training, onto-semiotic approach.

INTRODUCTION

The complex issue of making advances to clarify the nature of algebraic thinking is necessary from the point of view of education. As Radford says (2000, 238): "To go further, we want to add, we need to deepen our own understanding of the nature of algebraic thinking and the way it relates to generalization". The development of a comprehensive model of elementary algebra could facilitate the design of instructional activities that promote the emergence and progressive consolidation of algebraic reasoning.

In this report we address this problem by using some theoretical tools of the *Onto-semiotic approach* to research in mathematics education (Godino, Batanero and Font, 2007). We believe, together with various authors (Mason and Pimm, 1984; Carraher, Martinez and Schliemann, 2008; Cooper and Warren, 2008), that generalization and also the means to symbolize both generalization situations and modelling (in particular, using equations) are key features of algebraic reasoning.

First we summarize the vision of elementary algebra according to the onto-semiotic approach developed in Godino, Castro, Ake and Wilhelmi (2012); then we define two levels of proto-algebraic reasoning framed between two other levels: one, in which the reasoning is purely arithmetic (level 0 of algebraization), another in which the algebraic features are consolidated (level 3). Finally we highlight some implications of the model for the training of primary school teachers.

ONTO-SEMIOTIC APPROACH TO ELEMENTARY ALGEBRA

The pragmatic, anthropological and semiotics perspective of the onto-semiotic approachto research in mathematics education (OSA) (Godino, Batanero and Font, 2007; Godino, Font, Wilhelmi and Lurduy, 2011) provides theoretical tools that can help to characterize algebraic reasoning in terms of types of objects and processes involved in mathematical practice. Considering a mathematical practice as intrinsically algebraic can be based on the presence of certain types of objects and processes, usually considered in the literature as algebraic.

Prototypical algebraic objects

In the framework of elementary algebra the following are considered as prototypical algebraic objects:

1) Binary relations ─equivalence or order─ and their respective properties (reflexive, transitive and symmetric or antisymmetric). These relationships are used to define new mathematical concepts.

2) Operations and their properties, performed over elements of various sets of objects (numbers, geometric transformations, etc.). The algebraic calculation is characterized by the application of properties. Concepts like equation, inequality, and procedures such as elimination, factorization, etc. can also intervene.

3) Algebraic functions, generated by addition, subtraction, multiplication, division, potentiation and root extraction of the independent variable. It is necessary to consider different types of functions (polynomial, rational, radical) and its associated algebra (operations and properties).

4) Structures and their types (semigroup, group, ring,…) studied in abstract algebra.

Prototypical algebraic processes

Particularization and generalization processes are particularly importantfor algebraic activity, given the role of generalization as one of the key features of algebraic reasoning. Thus, for analysing algebraization levels of mathematical activity it is useful to focus attention on the objects resulting from the generalization and particularization processes. As a result of a generalization process we obtain a type of mathematical object we call *intensive object*, which becomes the rule that generates the class (collection or set) of generalised objects and that enables the identification of particular element as representative of the class (Godino et al., 2011). Through particularization processes new objects are obtained that we call *extensive* (particular) objects. A finite set or collection of particular objects simply listed should not be considered as an intensive until the subject shows the rule applied to delimit the constituent elements of the set. Then the set becomes something new, different from the constituent elements, as a unitary entity emerging from the set. Therefore, besides the generalization process giving rise to the set, there is a process of *unitization*.

Moreover, the new unitary entity has to be made ostensive or materialized by a name, icon, gesture or symbol. The *ostensive object* embodying the unitary object emerging from generalization is another object that refers to the new intensive entity, so there is a process of *representation* accompanying to the generalization and materialization processes. Finally, the symbol is released from the object which represents to become the object upon which actions are performed (*reification* process).

The different types of algebraic objects and processes can be expressed with different languages, preferably alphanumeric at higher levels of algebraization. Nevertheless, primary school pupils might also use other means of expression to represent objects and processes of algebraic nature (Radford, 2003).

In the next section, we describe the boundary between arithmetic and algebra in terms of the dualities and processes described. This boundary is not objective or platonically established, since these dualities and processes are relative to the context where mathematical practice is developed. In fact, the algebraic character is essentially linked to the subject's recognition of the rule that shapes the intensive, the consideration of the generality as a new unitary entity and its enactment by any semiotic register for subsequent analytical treatment. This threefold process (recognition or inference of generality, unitization and materialization) allowsdefining two primary levels of algebraic thinking, distinguishable from a more advanced level in which the intensive object is seen as a new entity represented with alphanumeric language.

ALGEBRAIZATION LEVELS

In this section we describe the characteristics of the practices to solve mathematical tasks, affordable in primary education, which allow to define different levels of algebraization. We propose to distinguish two *proto-algebraic* levels of primary algebraization. These levels are framed between a 0 level of algebraization and a third level in which mathematical activity can be considered as properly algebraic. This level is assigned, not to the task itself but to the mathematical activity that is performed. To explain the features of the algebraization levels we use examples of student teachers' responses to a task on geometric patterns. The description of such teaching experience is not the aim of this report due to space restrictions.

The problem posed to a sample of 52 student teachers is as follows:
See the following figure, and answers:

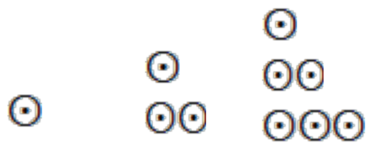


Fig. 1 Fig. 2 Fig. 3 ···

a) How many balls are there in figures on fourth and fifth position?

b) How many balls are there in figure 100?

Level 0 of algebraization

If we want to train primary school teachers so they can help their pupils to develope algebraic reasoning, we need to describe the mathematical practices of level 0, that is, those that do not include algebraic features. This is an unclear issue in the literature on early algebra (Carraher and Schliemann, 2007). We propose the following rule to assign level 0 of algebraization to a mathematical practice:

Extensive objects, expressed by natural, numerical, iconic or gestural language, are involved. Symbols that refer to an unknown value can also intervene, but that value is obtained as a result of operations on particular objects.

Figure 1 shows an example of mathematical activity we consider indicative of absence of algebraic thinking.

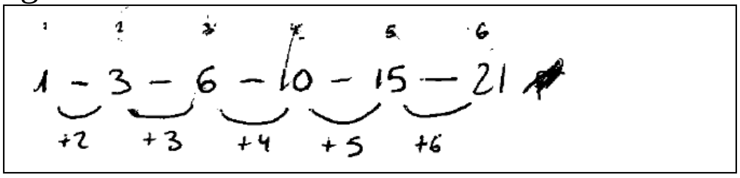


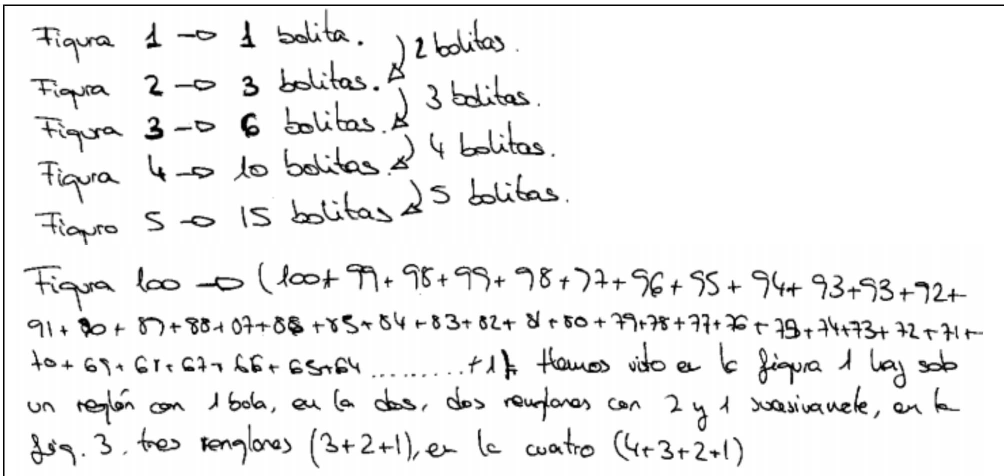
Figure 1. Level 0 response

The student writes the first six values of the independent variable of the function (order number of the figure) and below the number of balls that corresponds to each value, along with the criteria for obtaining these values (sum of successive natural numbers). He uses a numerical and visual language to express particular values, and makes no attempt to generalize the assignment criteria, or the initial and final sets of the correspondence. It is true that for the first six terms the student writes a formation rule, which extrapolated to any subsequent term would be indicative of the kind of factual generalization that Radford (2003) describes, but in this student's case such generalization does not occur.

Level 1 of algebraization

Intensive objects, whose generality is explicitly recognized by natural, numerical, iconic or gestural languages, are involved. Symbols that refer to the recognized intensive objectsare used, but there is no operation with those objects. In structural tasks relationships and properties of operations are applied and symbolically expressed unknown data may be involved.

Figure 2 shows a student's response that exemplifies this proto-algebraic level of thinking.



Translation: We have seen in Figure 1 there is a row with only one ball, in the second, two rows with 2 and 1 successively, in Fig. 3, three rows (3 +2 +1), in the fourth (4 +3 +2 +1)

Figure 2. Level 1 response

This student finds a general rule (*intensive object*) that allows him find the value of the function for any value of the independent variable (figure position) and that explicitly define with a sum of consecutive numbers. He uses ordinary language (to explain the formation rule) and arithmetic language (natural numbers and the sum of the first 100 natural numbers), but he has not been able to find a symbolic expression for this sum. The student can find the number of balls in figure 100, without forming this figure and without, therefore, explicitly count the beads, but operating with the sequence of particular numbers. It is a *factual generalization* (Radford, 2003). The operational scheme is limited to the concrete level, which however would allowhim to deal successfully with virtually any term.

Level 2 of algebraization

Indeterminate or variables expressed in literal-symbolic language to refer the intensive objects recognizedare involved, but they are linked to the spatial or temporal information of the context. In structural tasks the equations have the form Ax ± B = C. In functional tasks the generality is recognized, but there is no operation with variables to obtain canonical forms of expression.

An example of this algebraization level is shown in figure 3.

Al multiplicar una fila de bolitas por otra (a la que restamos 1 para no contar varias veces las mismas bolitas) obtenemos un cuadrado de bolitas ( ). Dividiéndolo entre 2 obtenemos un triángulo, pero aún habría que añadirle la fila nueva de esa serie para obtener la cantidad correcta.

Para este patrón se añaden tantas bolitas como indique el ordinal de la figura. Así, para la Fig 11, habrá que añadirle 11 bolitas a la cantidad que tuviese la Fig. 10.

$\boxed{x = \frac{n\cdot(n-1)}{2} + n}$ (n = ordinal de la figura)

Fig. 4 = $\frac{4\cdot 3}{2} + 4 = 6 + 4 =$ 10 bolitas

Fig 5 = $\frac{5\cdot 4}{2} + 5 =$ 15 bolitas

Fig. 100 = $\frac{100\cdot 99}{2} + 100 = \frac{9900}{2} + 100 = 4950 + 100 =$ 5050 bolitas

Translation:
Multiplying a row of balls by other (to which we subtract 1 not to count several times the same balls) we get a square of balls. … Dividing it by 2 we get a triangle, but still the new row of that series should be added to get the right amount.
For this pattern as many balls as those indicated by the ordinal of the figure are added. Thus, for Fig. 11, 11 balls will be added to the amount that Fig. 10 had.

Figure 3. Level 2 response

The student finds a correct formula for calculating the number of balls on the figure in any position, expressed with alphanumeric language. The justification of the formula is based on visual reasoning, expressed with confuse and not entirely correct natural language, since the visual inference of the formula requires forming a rectangle of sides *n*(*n*–1), and not a square. He does not operate with variables to get a canonical expression of the correspondence criterion. The student's reasoning includes aspects of contextual and symbolic generalizations (Radford, 2003). There is an explicit use of generic elements for the figure position and the corresponding number of balls, expressed in contextual terms and also symbolically. However, the mere use of literal symbols in a general expression is not enough to recognize the presence of aproperly algebraic practice.

Level 3 of algebraization

Intensive objects are generated which are literal-symbolically represented, and operations are carried out with them; transformations are made in form of symbolic expressions preserving equivalence. Operations are performed on the unknowns to solve equations of the form $Ax \pm B = Cx \pm D$, and symbolic and decontextualized canonical rules of expression of patterns and functions are formulated.

Level 3 of algebraization supposes, in our proposal, operate with the intensive objects symbolically represented, and therefore those objects have any contextual connotations. On the student's response (Figure 3) the symbolic expression of the proposed formula, $x = \frac{n(n-1)}{2} + n$, is related to the visual arrangement of the beads. Any attempt that the student could perform, operating with this expression to obtain alternative forms, for example, $f(n) = \frac{n(n+1)}{2}$, would be indicative of a more consolidated algebraic activity (level 3).

SUMMARY AND IMPLICATIONS FOR TEACHER TRAINING

We can identify more advanced levels of algebraic reasoning, such as those involving recognition, statement and justification of structural properties of mathematical objects involved. However, our approach focuses on identifying "what is algebraic" regarding "what is non-algebraic" in mathematical practice. In order to achieve this identification, we consider useful to introduce two intermediate levels of proto-algebraic activity.

We should recognize that boundaries between levels might sometimes be blurred and that within each level we can make distinctions that could lead to propose new levels of algebraization. However, our approach can be useful to guide the action of an elementary school teacher who tries to stimulate the progression of his/her pupils' mathematical thinking into progressive levels of generalization, representation and operative efficiency.

In figure 4 we summarize the main features of the proto-algebraic reasoning model we have described. In summary we propose to use three criteria to distinguish levels of elementary algebraic reasoning:

1. The presence of intensive algebraic objects (i.e., entities which have a character of generality, or indeterminacy).
2. Type of language used.
3. The treatment that is applied to these objects (operations, transformations) based on the application of structural properties.

The algebraization levels we propose are related to two aspects that Kaput (2008) identifies as characteristic of algebra and algebraic reasoning, namely algebra as:

a) Systematic symbolization of generalizations of regularities and constraints.
b) Syntactically guided reasoning and actions on generalizations expressed in conventional symbolic systems.

Aspect a) is specified in our model in levels 1 and 2 of proto-algebraic reasoning, while b) is associated with level 3, where algebra is already consolidated. Our requirement of using literal-symbolic language to assign a properly algebraic level (level 3) to mathematical practice, and the requirement of operate analytically/ syntactically with this language is concordant with other authors interested in defining "the algebraic" (e.g., Puig and Rojano, 2004).

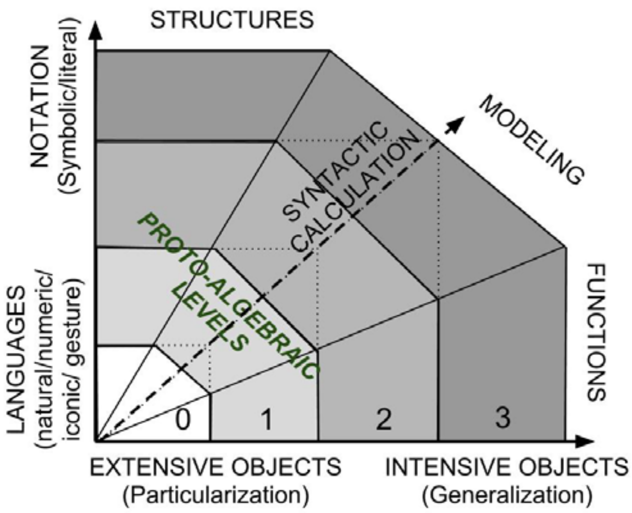


Figure 4. Levels of proto-algebraic mathematical thinking

In line with the proposals of the authors researching in the field known as "early algebra" (Carraher and Schliemann, 2007), we proposed to distinguish two primary levels of proto-algebraic reasoning to distinguish them from other forms stable or consolidated of algebraic reasoning. The key idea is to "make explicit the generality", of relations (equivalence or order), structures, rules, functions or on modelling mathematical or extra-mathematical situations, while operating with such generality.

The analysis of the nature of algebraic thinking has implications for teacher education. It is not enough to develop curriculum proposals (NCTM, 2000) that include algebra from the earliest levels of education; the teacher is required to act as the main agent of change in the introduction and development of algebraic thinking in elementary classrooms. The characterization model of "early algebra" that is proposed on this report, including the distinction of levels 1 and 2 of proto-algebraic reasoning, can be useful in training primary school teachers.

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