DIDACTIC EFFECTIVENESS OF EQUIVALENT DEFINITIONS OF A MATHEMATICAL NOTION

THE CASE OF THE ABSOLUTE VALUE[1]

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ABSTRACT

Quite often a mathematical object may be introduced by a set of equivalent definitions. One fundamental question consists of determining the "didactic effectiveness" of the techniques associated with these definitions for solving one kind of problem; this effectiveness is evaluated by taking into account the epistemic, cognitive and instructional dimensions of the study processes. So as to provide an example of this process, in this article we study the didactic effectiveness of techniques associated with different definitions of the absolute value notion (AVN). The teaching and learning of the AVN are problematic; this is proved by the amount and heterogeneity of the research papers that have been published. We propose a "global" study from an ontological and semiotic point of view (Godino, 2002; Wilhelmi, Godino and Lacasta, 2004).

1. MATHEMATICAL EQUIVALENCE *VS.* DIDACTIC EQUIVALENCE OF DEFINITIONS

One of the goals for the teaching of mathematics should be to channel everyday thinking habits towards a more technical-scientific form of thinking at an earlier stage, as a means for overcoming the conflicts between the (formal) structure of mathematics and the cognitive progress. The process of definition of mathematical objects represents "more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (Vinner, 1991, p.65). This fact justifies the great number of papers in the didactics of mathematics for which the subject matter is mathematical definition (Linchevsky, Vinner & Karsenty, 1992; Mariotti & Fischbein, 1997; De Villiers, 1998; Winicki-Landman & Leikin, 2000; etc.). We are interested in justifying the fact that the mathematical equivalence of two definitions of the same object does not imply their epistemic, cognitive or instructional equivalence, that is to say, the didactic equivalence.

From the viewpoint of the didactics of mathematics, one fundamental question consists of determining the *didactic effectiveness* of problem-solving techniques

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associated with a mathematical definition; this effectiveness is assessed by taking into account the *epistemic* (field of applicability of the techniques and mathematical objects involved), *cognitive* (effectiveness and cost in the use of the techniques by the individuals) and *instructional* (amount of material resources and time required for its teaching) dimensions. Hence, with the expression *didactic effectiveness* we refer to the articulation of these partial types of effectiveness in an educational project.

In relation to a mathematical notion it is necessary to: 1) determine mathematically equivalent definitions of the said notion; 2) describe the relations that are established between these definitions; 3) construct an explicit reference for the notion defined that envisages the complexity of objects and meanings that constitute the equivalent definitions associated with that notion in the different contexts of use; and 4) assess the didactic effectiveness of the techniques associated with the different mathematical definitions. A study of this kind may be performed for any kind of mathematical notion; however, the specific didactic decisions are consubstantial to each mathematical notion. In this article we aim to identify mathematically equivalent definitions of the notion of absolute value and discuss its equivalence or its diversity from a cognitive and instructional viewpoint. To do so, we answer the following questions:

- Is there a technique that minimises the cognitive and instructional cost of use of resources, that maximises the effectiveness of the individuals in the specific field of problems and that facilitates adaptation to new problems?
- Is it possible to classify the techniques according to their *scope or generality* (field of applicability), their *mutual implication* (one technique may be obtained deductively from another one) or their *role within the institutional practices* (social, cultural, conventional)?

So as to answer these questions it is necessary, in the first place, to determine the nature of the notion of absolute value and accept the complexity of objects and meanings that explicitly refer to it. In section 2, a set of research problems are described, the purpose of which is the understanding of the difficulties for the teaching and the learning of the AVN. From these investigations we deduce the ontological and semiotic complexity of the AVN, but none of them deals with the problem that arises when trying to integrate the meanings attributed to this notion in the different contexts of use. In section 4, we clarify a way to structure the models and meanings associated with the AVN and we describe its “overall” meaning. Beforehand, in section 3, we introduce the different definitions of the AVN and, backed by the calculation of the solutions of a linear equation with an absolute value, we indicate how these definitions condition mathematical practices.

2. NATURE OF THE NOTION OF ABSOLUTE VALUE

The teaching and learning of the AVN are problematic. This is proved by the amount and heterogeneity of the research papers that have been published. Gagatsis and Thomaidis (1994), after showing a succinct anthropology of the knowledge about “absolute value”, determine the processes for adapting that knowledge in Greek schools and interpret the students’ errors in terms of *epistemological obstacles* (linked to the historical study) and *didactic obstacles* (related with the processes of transposition). More recently, Gagatsis (2003, p.61) reasons from empirical data that the “obstacles encountered in the historical development of the concept of absolute value are evident in the development of students’ conceptions”.

From a professional point of view, Arcidiacono (1983) justifies a instruction of the AVN based on the graphic analysis on the Cartesian plane of piece-wise linear functions and Horak (1994) establishes that graphic calculators represent a more effective instrument than pencil and paper for performing this teaching. On the other hand, Chiarugi, Fracassina & Furinghetti (1990) carried out a study on the cognitive dimension of different groups of students faced with solving problems that involve the AVN. The study determines the need for research that will allow the errors and *misconceptions* to be overcome. On her part, Perrin-Glorian (1995) establishes certain guidelines for the institutionalisation of knowledge about the AVN in arithmetical and algebraic contexts; so she argues that the central function of the teacher’s didactic decisions in the construction of the AVN, that must take into account the students’ cognitive restrictions and must highlight the instrumental role of the AVN.

All these research papers implicitly consider that the nature of the AVN is transparent. From an ontological and semiotic point of view of mathematical cognition and instruction (Godino, 2002; Godino, Batanero and Roa, in print) it is necessary to theorise the notion of meaning in didactics. This theorising is done using the notion of semiotic function and an associated mathematical ontology. They start off with the elements of the technological discourse (notions, propositions, etc.) and it is concluded that its nature is inseparable from the pertinent systems of practices and contexts of use.

Godino (2002) identifies the “system of practices” with the contents that an institution assigns to a mathematical object. The description of the meaning of reference for an object is presented as a list of objects classified into six categories: problems, actions, language, notions, properties and arguments. Wilhelmi, Godino and Lacasta (2004) argue in what way this description of the system of practices is insufficient for the description of the institutional meanings of reference and, in order to overcome those deficiencies, the theoretical notions of *model* and of *holistic meaning* of a mathematical notion are introduced. These notions will allow us to structure the different definitions of the AVN and the description of the meaning of the AVN as a “whole”, in a coherent complex whilst drawing some conclusions of a macro and micro didactic nature.

3. DEFINITIONS FOR THE NOTION OF ABSOLUTE VALUE

In this section, we introduce some definitions of the AVN associated with different contexts of use and we briefly indicate how these definitions, as objects emerging from the different subsystems of practices, condition the operational and discursive rules. In the arithmetical context, the AVN represents a rule that "leaves the positive numbers unchanged and turns the negative numbers into positive ones".

> "The absolute value of x, denoted by $|x|$, is defined as follows:
>
> $$|x| = x \text{ if } x > 0; \qquad |x| = -x \text{ if } x < 0; \qquad |0| = 0$$
>
> Thus, the absolute value of a positive number or zero is equal to the number itself. The absolute value of a negative number is the corresponding positive number, since the negative of a negative number is positive." (Leithold, 1968, p.10).

The absolute value provides the set of real numbers with a metric; the distance of a real number x to the origin 0 is defined by the relation: $d(x; 0)=|x|$.

> "Intuitively, the absolute value of a represents the distance between 0 and a, but in fact we will *define* the idea of 'distance' in terms of the 'absolute value', which in turn was defined in terms of the ordering." (Ross, 1980, p.16).

In the geometrical context, the NVA may be understood in terms of vectors as the module for a one-dimensional vector. What is more, this fact may be generalised as a property that is derived from the "ordered" and "complete" nature of **R** (Aliprantis & Burkinshaw, 1998, p.66–67).

The classic definition of absolute value, as a basic notion for the foundations of mathematical analysis, is sometimes reformulated in terms of the maximum function: $|x| = \max\{x; -x\}$. In this same context, the AVN is often introduced using a piece-wise function in **Q** and, by extension, in **R**.

> "For any rational number q: $|q| = \begin{cases} q & \text{si } q \geq 0 \\ -q & \text{si } q < 0 \end{cases}$ [...] We extend the definition of 'absolute value' from **Q** to **R** [...] $|x|$ equal x if $x \geq 0$, and $-x$ if $x < 0$." (Truss, 1997, pp.70–102).

Finally, it is easy to demonstrate that: $|x| = +\sqrt{x^2}$ (Mollin, 1998, p.47).

The aforementioned definitions are mathematically equivalent, but their use conditions mathematical activity: they do not involve the same mathematical objects in the resolution of a same problem. For example, let it be the linear equation with absolute value $|x - 2| = 1$, its solution in an arithmetical context involves a reasoning of the kind: "the absolute value of a number is 1, then this number is 1 or –1; What number, when subtracting 2 from it, gives 1?, What number, when subtracting 2 from it, gives –1?". The formalisation of this method may be done in the following way:

$$|x-2| = 1 \Rightarrow \begin{cases} x - 2 = 1 \Rightarrow x = 3 \\ x - 2 = -1 \Rightarrow x = 1 \end{cases}$$

However, the analytical demonstration, according to the compound function definition, is performed in the following way:

$$|x-2|=1 \quad \Rightarrow \sqrt{(x-2)^2}=1 \Rightarrow (x-2)^2=1 \Rightarrow x^2-4x+3=0 \Rightarrow$$
$$\Rightarrow x=\frac{4\pm\sqrt{16-12}}{2}=\frac{4\pm 2}{2}=\begin{cases} x=3 \\ x=1 \end{cases}$$

Next, in section 4.1, we shall show the onto-semiotic complexity of the AVN, that is deduced from the diversity of contexts of use, from the definitions associated with them and the operational and discursive practices that these definitions condition and, in section 4.2, backed by the theoretical notion of *holistic meaning* (Wilhelmi, Godino and Lacasta, 2004), we shall organise the models of absolute value, whilst showing the relations that are established between them.

4. ONTO-SEMIOTIC COMPLEXITY OF THE ABSOLUTE VALUE

4.1. STRUCTURE OF DEFINITIONS, MODELS AND MEANINGS ASSOCIATED WITH THE NOTION OF ABSOLUTE VALUE

The professional mathematician identifies the same formal structure in the variety of objects and (operational and discursive) practices; a structure that he/she considers to be "the mathematical object". This formal structure represents the implicit reference in the resolution of types of problems associated with the variety of systems of practices and objects emerging in the different contexts of use. Figure 1 shows schematically the diversity of objects associated with the AVN.

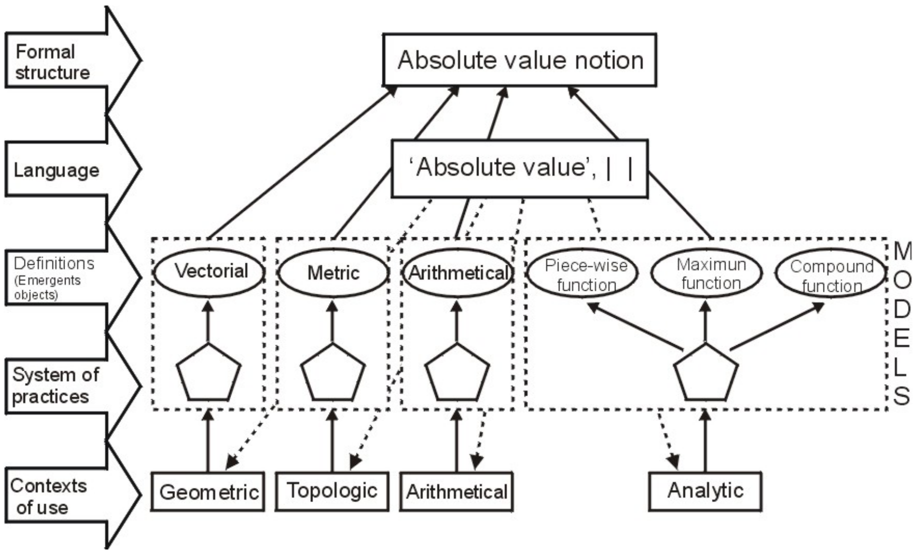

Figure 1. Structure for the models and meanings associated with the absolute value.

Each definition represents an object emerging from a system of practices in a given context of use. No definition may be privileged *a priori*. Each "emergent object - system of practices" binomial determines a *model* of the AVN. The model is then a coherent form for structuring the different contexts of use, the mathematical practices relating to them and the objects emerging from such practices; so forming a network or *local epistemic configuration* (associated with a specific context of use).

4.2. HOLISTIC MEANING OF THE NOTION OF ABSOLUTE VALUE

From the strictly *formal and official* viewpoint (Brown, 1998), it is accepted that the definition of a mathematical object constitutes its meaning. The description of the system of models and meanings associated with a notion is obtained from the statement and demonstration of a theorem for characterisation: privilege of one of the definitions and justification of the equivalence of the rest of the definitions.

The empirical data provided by Leikin & Winicki-Landman (2000) allow to state that the equivalence of mathematical definitions cannot be assessed just from the epistemological viewpoint, it is necessary take into account the *cognitive* (What strategies for action generate each one of the definitions?), *instructional* (What definition is the most suitable within a given project for teaching?) and *didactic* (What relationship is established between the personal meaning learnt and the institutional meaning intended?) dimensions. The *holistic meaning* (Wilhelmi, Godino & Lacasta, 2004) comes from the coordination of the meaning attributed to the models associated with the notion of equality and the tensions, filiations and contradictions that are established between them.

5. COGNITIVE EFFECTIVENESS OF THE ARITHMETIC MODELS AND "PIECE-WISE FUNCTION" OF THE ABSOLUTE VALUE

As we mentioned earlier, from the viewpoint of the didactics of mathematics, a fundamental question consists of determining the *didactic effectiveness* of a mathematical process for problem-solving. In this section we aim to analyse the *cognitive* dimension (effectiveness and cost in the use of the techniques by individuals) of the problem-solving techniques associated with the "arithmetical" definitions and "piece-wise function". To do so, we use an experimental study with a group of 55 students (trainee teachers) solving a set of elemental exercises that require the AVN (Table 1).

1. Complete, if you can, the following equalities:

   $|-2| =$ $\quad |2| =$ $\quad |0| =$ $\quad |\sqrt{-2}\,| =$

   $|\sqrt{2}\,| =$ $\quad |-\sqrt{2}\,| =$ $\quad |2-\sqrt{2}\,| =$ $\quad |\sqrt{2}-2| =$

2. State, if you can, the numbers that would have to be inserted to replace the dots so the following expressions will be correct:

   $|\ldots - 2| = 1;$ $\quad |\ldots + 2| = 1;$ $\quad |\ldots - 2| = 0;$ $\quad |(\ldots)^2 - 4| = 0;$

   $|(\ldots)^2 + 4| = 0;$ $\quad |(\ldots)^2 - 1| = 1;$ $\quad |(\ldots)^2 - 3| = 1$

3. Represent in a graphic way the function $f(x) = |x+1|$.
4. Let a be a real number. Complete, if you can, the following equalities:

   $|-a| =$ $\quad |a| =$ $\quad |a-2| =$

   $|-a-2| =$ $\quad |2-a| =$ $\quad |a+2| =$

Table 1. Questionnaire.

5.1. PREDOMINANT MODEL AND EFFECTIVENESS IN PROBLEM-SOLVING

Generically, we affirm that a person understands the AVN if he/she is capable of distinguishing its different associated models, structuring the said models in a complex and coherent group and meeting the operative and discursive needs in relation to the AVN in the different contexts of use.

Formally, a definition may be reduced to axioms; however, in a process of study, the definition represents a formalization of a *pertinent* notion (it allows a consistent interpretation of a problem) or *operative* (it conditions a useful action). The only means for distinguishing the meaning attributed by an individual to an object is by means of a situation or a set of problems that may be solved by using different models capable of generating pertinent and useful actions, that, however, comply with different "economic" laws.

The experimental work performed has allowed us to classify the students according to the model of absolute value associated with the operative and discursive practices in relation to the problems proposed (that determines a certain level of effectiveness). So as to be able to classify the students, it is necessary to interrelate a collection of tasks and determine (with a level of approximation) the tasks that allow the performance of other tasks to be assured.

5.2. ANALYSIS OF A QUESTIONNAIRE

The main purpose of the experimentation is to empirically support the thesis according to which the models "arithmetical" and "piece-wise function" associated with the AVN are extremely similar (see Section 4.2). The analysis of the institutional meanings determines selection criteria of the variables for the *implicative* study (Gras, 1996). The system of variables is shown in Table 2.

Variable	Description	No. of answers
v1	$\lvert \pm\sqrt{2} \rvert = \sqrt{2}$ (without numerical approximation)	47
v2	$\lvert \pm\sqrt{2} \rvert \approx 1{,}41$ (with numerical approximation)	20
v3	$\lvert \sqrt{-2} \rvert =$ no $\exists$ (does not make sense in **R**)	16
v4	$\lvert 2-\sqrt{2} \rvert \approx 0{,}59$ (with numerical approximation)	30
v5	$\lvert 2-\sqrt{2} \rvert = 2-\sqrt{2}$ (without numerical approximation)	17
v6	$\lvert \sqrt{2}-2 \rvert = 2-\sqrt{2} \approx 0{,}59$ (with or without numerical approx.)	30
v7	Two solutions in |… – 2| = 1 or in |… +2| = 1	21
v8	Determination of the two solutions of $\lvert(\ldots)^2 - 4\rvert = 0$	26
v9	$\lvert(\ldots)^2 + 4\rvert = 0$ has no solution	24
v10	Solution of $\lvert(\ldots)^2 - 1\rvert = 1$: 0	29
v11	Solution of $\lvert(\ldots)^2 - 1\rvert = 1$: 0 and $\sqrt{2}$, $-\sqrt{2}$ or $\pm\sqrt{2}$	9
v12	At least two solutions for $\lvert(\ldots)^2 - 3\rvert = 1$	25
v13	They construct the graph and give the formula correctly	28
v14	They construct the graph and give the formula incorrectly	15
v15	At least 4 correct sections from exercise 4	14
v16	Mean in the course ≥ 14 (out of 20)	17

Table 2. Small set of variables.

The aim is to find whether, in the sample, the fact of having answered a question correctly statistically implies the response to another question. In particular, it is admissible to expect that any individual who is capable of performing a task that is more complex than another (and that generalises it in a certain way), then he/she will also be capable of performing the second one. However, this is not always so; in many circumstances it is necessary to compare certain hypotheses for implementing a hierarchy for performing tasks. Below, we comment on some of these implications:

- *Implication at* 99%. A group of students is stable in solving the equations: they perform the search for roots in an equation (linear and quadratic) with absolute value in a routine manner.
- *Implication at* 95%.
  - v2 → v4: Mostly, the students who have given an approximation for $|\sqrt{2}|$, also establish that $|2-\sqrt{2}| \approx 0{,}59$. One possible interpretation: the arithmetical model of absolute value is understood as a rule that operates on the "numbers", that is to say, numbers "in decimal format".
  - v3 → v9: The didactic contract assumed by the vast majority of the students establishes the existence of a solution for any problem; their function is find it (the sentence "when you can" is skipped by these students). Hence, the relation "v3 → v9" distinguishes a group of students that separates action and meaning.
  - v16 → v6 and v16 → v13. The students who have a "good" behaviour in the course mostly operate the absolute value ($|\sqrt{2}-2| = 2-\sqrt{2}$) "symbolically" and understand the $f(x) = |x + 1|$ function analytically and graphically.

A wider question that may be posed is whether the fact of having answered a set of questions correctly implies (in a preferential manner) the right answer in another set of questions. The *hierarchical analysis* (Gras, 1996) allows the implicative relationships between the kinds of questions to be described in a more "dynamic" way and, therefore, constitutes a response to the question posed. Based on the experimental data, it is established that the most significant classes are: v7 → v12 → v8 → v6 and v15 → v16 → v13. What individuals contribute to the formation of each one of the classes? The students who most contribute to both classes are those who perform the tasks symbolically and are capable of applying the model piece-wise function systematically and effectively.

6. MACRO AND MICRO DIDACTIC IMPLICATIONS

The cognitive difficulties (Chiarugi, Fracassina & Furinghetti, 1990) and the incapacity of the educational institution to draw up a pertinent curriculum for the introduction and development of the AVN (Perrin-Glorian, 1995; Gagatsis and Thomaidis, 1994) has led to merely technical teaching based on the arithmetical model (as a rule that "removes the minus sign"). The arithmetical model of the AVN proves to be a didactic obstacle that restricts, in many cases, the personal meaning to a mere game of symbols. This obstacle is shown in different ways; for example: $|a| = a$ and $|-a| = a$, for any $a \in \mathbf{R}$; $|\sqrt{2}-2| = \sqrt{2}+2$, etc.

Macrodidactic implications

The introduction of the absolute value in the arithmetical context represents an unfortunate decision in modern-day school institutions: it means the inclusion in the curriculum of the notion "absolute value" for merely cultural reasons. However, the curricular structure is not ready at the present to properly cope with the study of the notion in an exclusively arithmetical context. It would be advisable to "temporarily" remove the notion. This would be temporary, either until a pertinent didactic transposition, or until the students start to study the theory of functions, central in relation to the notion of absolute value (Arcidiacono, 1983; Horak, 1994).

This "drastic" didactic decision means, on the one hand, the acceptance by the educational institution of the existence of a didactic transposition that is not pertinent in relation to the notion of absolute value and, on the other hand, its incapacity to produce a *viable* (admissible cost of material and time resources), *reproducible* (institutional stability in relation to the availability of resources) and *reliable* (the personal meanings learnt are representative of the institutional meanings intended) "*de-transposition*" (Antibi and Brousseau, 2000). Gagatsis (2003, p.61) get a similar conclusion: "There are also a number of obstacles with didactic origin relating to the 'strange' didactic transposition or the restrictions of the educational system [...] There is a problem of legitimization of the content to be taught."

Microdidactic implications

From the point of view of learning, the models associated with mathematical notion are ordered according to their hierarchy. The structuring of the models is carried out in terms of the "field" of the latter in the curriculum. The dominant model must clearly and specifically participate in the first encounter with the notion. For the AVN, the model "piece-wise function", using the graphic representation of the function in the Cartesian plane and using the discursive practices pertaining to the theory of functions.

Hence, it is necessary to establish a didactic engineering for developing the "absolute value" object (understood as a system). This engineering will have to articulate the epistemological analysis with the methodological and time restrictions within each specific institution. In relation to the AVN, the objective consists of establishing a system of practices that will make the explicit interaction of the arithmetical model with the rest of the models possible and, most particularly, with the analytical model.

7. SYNTHESIS AND CONCLUSIONS

The notion of holistic meaning of a mathematical notion makes it possible to describe the latter as an epistemic configuration that takes into consideration both the praxis and discursive elements of mathematical activity. Furthermore, it provides an instrument for controlling and assessing the systems of practices implemented and an observable response (and, in a certain way, quantifiable) for the analysis of personal meanings. More precisely speaking:

- The notion of holistic meaning (network of models) represents the structuring of the knowledge targeted and may be used to determine the degree of representation of a system of practices implemented in relation to the institutional meaning intended.
- The notions of model and holistic meaning provide a response to the questions: What is a mathematical notion? What is understanding this notion?; in particular, What is the AVN? What does understanding the AVN mean?

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