On the deviation and the type of certain local Cohen-Macaulay rings and numerical semigroups

E. Kunz and R. Waldi

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

Abstract In J. Herzog and E. Kunz. On the deviation and the type of a Cohen-Macaulay ring. manuscripta math. 9 (1973) 383-388 it was shown that for any pair $(d, t) \in \mathbb{N} \times \mathbb{N}_+$ with $(d, t) \neq (1, 1)$ there exists a local Cohen-Macaulay ring R having deviation $d(R) = d$ and type $t(R) = t$. By E. Kunz. Almost complete intersection are not Gorenstein rings. Journal Alg. 28 (1974) 111-115 the case $d(R) = 1, t(R) = 1$ cannot occur. In this paper certain Cohen-Macaulay rings are studied for which there are close relations between deviation, type and embedding dimension. Similar relations for other classes of local rings have been proved in the recent paper by L. Sharifan. A class of Artinian local rings of homogeneous type. Bull. Iranian Math. Soc. 40 (2014) 157-181. Our relations will be applied to numerical semigroups (or equivalently monomial curves) and lead also to some further cases, generalizing E. Kunz. On the type of certain numerical semigroups and a question of Wilf. Semigroup Forum 93 (2016) 205-210 with ring-theoretic proofs, in which a question of H. Wilf. A circle-of-lights algorithm for the money-changing problem. Amer. Math. Monthly 85 (1978) 562-565 has a positive answer.

1. Introduction

Let R be a complete Noetherian local ring. Write $R = P/I$ where P is a regular local ring. If $\mu(I)$ denotes the minimal number of generators of I, then the number

$$d(R) := \mu(I) - (\dim(P) - \dim(R))$$

is independent of the chosen presentation of R and is called the *deviation* of R. If $\{z_1, \dots, z_b\}$ is a regular sequence in the maximal ideal $\mathfrak{m}_R$ of R, then $d(R) = d(R/(z_1, \dots, z_b))$ ([HK_1], 1.16 and 3.10b)).

If R is a Cohen-Macaulay ring and $\{z_1, \dots, z_b\}$ a maximal regular sequence in $\mathfrak{m}_R$, hence $b = \dim(R)$, then let $\mathrm{Soc}(R/(z_1, \dots, z_b))$ be the socle of $R/(z_1, \dots, z_b)$, i. e. the $R/\mathfrak{m}_R$-vector space of the elements in $R/(z_1, \dots, z_b)$ annihilated by the maximal ideal $\mathfrak{m}_R$. Then the number

$$t(R) := \dim_{R/\mathfrak{m}_R} \mathrm{Soc}(R/(z_1, \dots, z_b))$$

is independent of the maximal regular sequence and is called the *type* of R.

Informations about the deviation and the type can be found, for example in [HK_2].

In the following special rings R will be studied in which close relations between $d(R), t(R)$ and the embedding dimension $\mathrm{edim}(R)$ of R exist.

2. Relations between deviation and type

Let R be a complete local Cohen-Macaulay ring of dimension b and embedding dimension $a+b+1, a \neq 0$. Choose a minimal system $\{x_1, \dots, x_a, y, z_1, \dots, z_b\}$ of generators of $\mathfrak{m}_R$ such that $\{z_1, \dots, z_b\}$ is a regular sequence.

2.1 Theorem. a) If $x_i x_j \in (y, z_1, \dots, z_b)$ $(i, j = 1, \dots, a)$, then

$$d(R) \geq \binom{\mathrm{edim}(R) - \dim(R)}{2} - (\mathrm{edim}(R) - \dim(R))$$

and

$$t(R) \leq \mathrm{edim}(R) - \dim(R).$$

b) If even $x_i x_j \in (z_1, \dots, z_b)$ $(i, j = 1, \dots, a)$, then there exists $\delta_1 \in \{0, \dots, a\}$ such that

$$d(R) = \binom{\mathrm{edim}(R) - \dim(R)}{2} - \delta_1$$

and $\delta_2 \in \{0, 1\}$ such that

$$t(R) = \mathrm{edim}(R) - \dim(R) - \delta_2.$$

Moreover $\delta_2 = 0$ if and only if $\delta_1 = 0$.

c) If $x_i x_j \in (z_1, \dots, z_b)$ $(i, j = 1, \dots, a)$ and if there exists an element $\sigma \in \mathfrak{m}_R \setminus (x_1, \dots, x_a, z_1, \dots, z_b)$ with $\sigma \cdot (y, x_1, \dots, x_a) \subset (z_1, \dots, z_b)$, then $\delta_1 = \delta_2 = 0$.

Proof. By the remarks of the introduction it suffices to consider the 0-dimensional case. Then there is a complete regular local ring $(P, \mathfrak{n})$ with a regular system of parameters $\{X_1, \dots, X_a, Y\}$ and a presentation $R = P/I$ where $x_i = X_i + I$ $(i = 1, \dots, a), y = Y + I$. Moreover $I \subset \mathfrak{n}^2$ and $I \not\subset \mathfrak{n}^3$ since $a \neq 0$.

We shall show the following lemma on Artinian local rings. Similar relations as in part b) of the lemma have been shown by L. Sharifan [Sh] for a different class of Artinian local rings generalizing previous results of Elias and Valla [EV], Rossi and Valla [RV] and Sally [S].

Let $(P, \mathfrak{n})$ be a regular local ring of dimension $a+1$ and I an $\mathfrak{n}$-primary ideal of *initial degree* $s \geq 2$, that is $I \subset \mathfrak{n}^s, I \not\subset \mathfrak{n}^{s+1}$. The number s is an invariant of $R = P/I$. In the situation of the theorem we have $s = 2$.

2.2 Lemma. Let $\{X_1, \dots, X_a, Y\}$ be a regular system of parameters of P.
a) If $(X_1, \dots, X_a)^s \subset I + (Y)$, then

$$\binom{a+s-1}{s} \leq \mu(I) \text{ and } t(R) \leq \binom{a+s-1}{s-1}.$$

b) If even $(X_1, \dots, X_a)^s \subset I$, then

$$1 + \binom{a+s-1}{s} \leq \mu(I) \leq \binom{a+s}{s} \text{ and } \binom{a+s-2}{s-1} \leq t(R) \leq \binom{a+s-1}{s-1}.$$

Moreover $\mu(I) = \binom{a+s}{s}$ if and only if $t(R) = \binom{a+s-1}{s-1}$.
c) If $(X_1, \dots, X_a)^s \subset I$ and $\mathrm{Soc}(R) \not\subset (X_1, \dots, X_a)^{s-1} + I/I$, then

$$t(R) > \binom{a+s-2}{s-1}.$$

The case $s = 2$ of the lemma now implies Theorem 2.1 for $\dim(R) = 0$ which is sufficient to prove it in general. In fact, for Theorem 2.1a),b) and c) the corresponding conditions of the lemma are satisfied. The conclusions follow since here $a+1 = \mathrm{edim}(R)$, $s = 2$ and $\mu(I) = d(R) + \mathrm{edim}(R)$. In 2.1c) we have the assumption that there is a $\sigma \in \mathrm{Soc}(R) \setminus (x_1, \dots, x_a)$. By 2.2c) we obtain $t(R) > \mathrm{edim}(R) - 1$. Then $t(R) = \mathrm{edim}(R)$ and $d(R) = \binom{\mathrm{edim}(R)}{2}$ follow from 2.2b). $\square$

Proof of the lemma

a) $\overline{P} := P/(Y)$ is a regular local ring of dimension a with maximal ideal $\overline{\mathfrak{n}} := (X_1, \dots, X_a)\overline{P}$. Since $\mathfrak{n}^s \subset I + (Y) \subset \mathfrak{n}^s + (Y)$ the image of I in $\overline{P}$ is $\overline{\mathfrak{n}}^s$, hence

$$\mu(I) \geq \mu(\overline{\mathfrak{n}}^s) = \binom{a+s-1}{s}.$$

Further $R/YR = \overline{P}/\overline{\mathfrak{n}}^s$ and $\mathrm{length}(R/YR) = \binom{a+s-1}{s-1}$. Since $\mathrm{Soc}(R) \subset (0:Y)_R$ the exact sequence

$$0 \to (0:Y)_R \to R \xrightarrow{Y} R \to R/YR \to 0$$

implies that

$$t(R) \leq \mathrm{length}((0:Y)_R) = \mathrm{length}(R/YR) = \binom{a+s-1}{s-1}. \tag{1}$$

b) For $\mathfrak{P} := (X_1, \dots, X_a)P$ we have $\mathfrak{P}^s \subset I$, hence as in a) $\mathfrak{n}^s \subset I + (Y)$. Since the ideal $\mathfrak{P}$ is generated by a regular sequence the ring $S := P/\mathfrak{P}^s$ is Cohen-Macaulay ([M], Exercise 17.4) and Y is a non-zero-divisor of S. Let $\mathfrak{m}_S$ be the maximal ideal of S and $J \subset \mathfrak{m}_S$ the kernel of the epimorphism $S \to R$. Then

$(0) \neq J \subset \mathfrak{m}_S^s = Y\mathfrak{m}_S^{s-1}$, hence there is a unique ideal $J' \subset \mathfrak{m}_S^{s-1}$ such that $J = YJ'$. It follows that $\mathrm{Soc}(R) \subset (0:Y)_R = J'/J$, and (1) implies

$$t(R) \leq \mathrm{length}(J'/J) = \binom{a+s-1}{s-1}. \tag{2}$$

Equality holds if and only if $\mathrm{Soc}(R) = J'/J$, i.e. $\mathfrak{m}_S J' = J$.

Further, since J and J' are isomorphic S-modules, we have by (2)

$$1 \leq \mu(J) = \mathrm{length}(J'/\mathfrak{m}_S J') \leq \mathrm{length}(J'/YJ') = \binom{a+s-1}{s-1}. \tag{3}$$

As above $\mu(J) = \binom{a+s-1}{s-1}$ if and only if $\mathfrak{m}_S J' = J$.

Adding preimages in P of a minimal system of generators of J to the set $\{X_1^{\alpha_1}\cdots X_a^{\alpha_a} | \sum \alpha_i = s\}$ we get a minimal system of generators of I, consequently $\mu(I) = \mu(J) + \binom{a+s-1}{s}$ and by (3)

$$1 + \binom{a+s-1}{s} \leq \mu(I) \leq \binom{a+s-1}{s-1} + \binom{a+s-1}{s} = \binom{a+s}{s}.$$

In particular we have $\mu(I) = \binom{a+s}{s}$ if and only if $\mu(J) = \binom{a+s-1}{s-1}$, i.e. $\mathfrak{m}_S J' = J$, and it follows that $\mu(I) = \binom{a+s}{s}$ if and only if $t(R) = \binom{a+s-1}{s-1}$.

It remains to be shown that $t(R) \geq \binom{a+s-2}{s-1}$. For this we consider the discrete valuation ring $V := P/\mathfrak{P} = S/\mathfrak{p}$ ($\mathfrak{p} := \mathfrak{P}/\mathfrak{P}^s$) with the prime element Y. As $\mathfrak{P}$ is generated by a regular sequence $\mathrm{gr}_{\mathfrak{P}}(P)$ is a polynomial ring in a variables over V, in particular

$$\mathfrak{p}^{s-1} = \mathfrak{P}^{s-1}/\mathfrak{P}^s \text{ is a free } V-\text{module of rank } \binom{a+s-2}{s-1}. \tag{4}$$

Since R is Artinian there is a $\lambda \in \mathbb{N}$ such that $Y^\lambda \in J \subset J'$, hence $Y^\lambda \mathfrak{p}^{s-1} \subset J' \cap \mathfrak{p}^{s-1} \subset \mathfrak{p}^{s-1}$, where $\mathfrak{p}^{s-1}$ and $Y\mathfrak{p}^{s-1}$ are isomorphic V-modules. Hence $J' \cap \mathfrak{p}^{s-1}$ is free of rank $\binom{a+s-2}{s-1}$ as well.

Further $\mathfrak{m}_S = \mathfrak{p} + YS$, $\mathfrak{p}\mathfrak{p}^{s-1} = 0$ and $S/\mathfrak{p}^{s-1} \cong P/\mathfrak{P}^{s-1}$ is a Cohen-Macaulay ring ([M], loc. cit.) with non-zero-divisor Y. This implies

$$\mathfrak{m}_S(J' \cap \mathfrak{p}^{s-1}) = Y(J' \cap \mathfrak{p}^{s-1}) = (YJ') \cap \mathfrak{p}^{s-1} = J \cap \mathfrak{p}^{s-1} \subset J$$

hence

$$J' \cap \mathfrak{p}^{s-1}/Y(J' \cap \mathfrak{p}^{s-1}) = J' \cap \mathfrak{p}^{s-1}/J \cap \mathfrak{p}^{s-1} \cong J' \cap \mathfrak{p}^{s-1} + J/J \subset \mathrm{Soc}(R) \tag{5}$$

and finally

$$t(R) \geq \mathrm{length}(J' \cap \mathfrak{p}^{s-1})/Y(J' \cap \mathfrak{p}^{s-1}) = \mathrm{rank}_V(J' \cap \mathfrak{p}^{s-1}) = \binom{a+s-2}{s-1}. \tag{6}$$

c) If in addition $t(R) = \binom{a+s-2}{s-1}$, then by (5) and (6)

$$\operatorname{Soc}(R) = J' \cap \mathfrak{p}^{s-1} + J/J \subset (X_1, \ldots, X_a)^{s-1} + I/I,$$

which proves assertion c) of the lemma. $\square$

2.3 Corollary. Under the assumptions of theorem 2.1 let $x_i x_j \in (z_1, \ldots, z_b)$ $(i, j = 1, \ldots, a)$ and $x_i y \in (z_1, \ldots, z_b)$ $(i = 1, \ldots, a)$. Then

$$d(R) = \binom{\operatorname{edim}(R) - \dim(R)}{2} \text{ and } t(R) = \operatorname{edim}(R) - \dim(R).$$

Proof. Apply 2.1c) with $\sigma := y^\mu$ where μ is the largest number with $y^\mu \notin (x_1, \ldots, x_a, z_1, \ldots, z_b)$. $\square$

3. Application to numerical semigroups

Let H be a numerical semigroup and $\{h_1, \ldots, h_e\}$ a system of generators of H. The completed semigroup algebra of H over a field K is defined as

$$K[[H]] = K[[t^{h_1}, \ldots, t^{h_e}]] \subset K[[t]]$$

with a variable t. The deviation and the type of H can be defined to be those of $K[[H]]$. The deviation of H can also be expressed by the relations between the generators of H (see [H]), and the type as the number of the pseudo-Frobenius numbers of H. Theorem 2.1 can be applied to $K[[H]]$ for certain numerical semigroups H and implies results about the deviation, the type and the embedding dimension of these H, generalizing those of [K_2] about the type and giving ring-theoretic proofs.

Assume that $H \neq \mathbb{N}$ is a numerical semigroup with minimal system of generators E. Let $p \neq q$ elements from E (not necessarily coprime) and let $E = \{p, q, h_1, \ldots, h_{e-2}\}$ $(e := \operatorname{edim}(H))$.

3.1 Corollary. a) If $h_i + h_j \in (p + H) \cup (q + H)$ $(i, j = 1, \ldots, e-2)$, then

$$d(H) \geq \binom{\operatorname{edim}(H) - 1}{2} - (\operatorname{edim}(H) - 1)$$

and

$$t(H) \leq \operatorname{edim}(H) - 1.$$

b) If even $h_i + h_j \in p + H$ $(i, j = 1, \ldots, e-2)$, then there exists $\delta_1 \in \{0, \ldots, e-2\}$ such that

$$d(H) = \binom{\operatorname{edim}(H) - 1}{2} - \delta_1$$

and $\delta_2 \in \{0, 1\}$ such that

$$t(H) = \operatorname{edim}(H) - 1 - \delta_2.$$

Moreover $\delta_2 = 0$ if and only if $\delta_1 = 0$.

c) If $h_i + h_j \in p + H$ $(i,j = 1,\ldots,e-2)$ and if there exists $h \in H \setminus \{0\}$ with $h \notin p + H, h \notin h_i + H$ $(i = 1,\ldots,e-2)$ such that $h + q \in p + H$ and $h + h_i \in p + H$ $(i = 1,\ldots,e-2)$, then $\delta_1 = \delta_2 = 0$.

d) If $h_i + h_j \in p + H$ $(i,j = 1,\ldots,e-2)$ and $h_i + q \in p + H$ $(i = 1,\ldots,e-2)$, then $\delta_1 = \delta_2 = 0$.

Assertions b), c) and d) remain true, if p and q are exchanged.

Proof. With $z := t^p, y := t^q, x_i := t^{h_i}$ $(i = 1,\ldots,e-2)$ we have

$$K[[H]] = K[[x_1,\ldots,x_{e-2},y,z]]$$

and this is a one-dimensional local Cohen-Macaulay algebra.
a) From $h_i + h_j \in (p+H) \cup (q+H)$ it follows that $x_i x_j \in (y,z)$ $(i,j = 1,\ldots,e-2)$, and we are in the situation of part a) of Theorem 2.1. The assertions about the invariants of the semigroups follow.

The assumptions of b),c) correspond to those of Theorem 2.1, if we set $s = t^h$ in c), and those of d) to the assumptions of Corollary 2.3. The statements for semigroups follow directly. $\square$

Wilf [W] has asked whether the relation

$$\mathrm{edim}(H) \cdot (c(H) - g(H)) \geq c(H)$$

holds for all numerical semigroups H where $c(H)$ denotes the conductor and $g(H)$ the genus of H. In [FGH], Theorem 20 the relation

$$(t(H) + 1)(c(H) - g(H)) \geq c(H)$$

was proved. Since $t(H) \leq \mathrm{edim}(H) - 1$ in all cases of the corollary it follows that Wilf's question has a positive answer in the situations described there.

4. Special cases

Corollary 3.1 will now be applied to certain more explicitly described semigroups. Let E be the minimal system of generators of a numerical semigroup H.

4.1 Example. If at most two not necessarily coprime elements p, q of E are smaller than the Frobenius number of H, then

$$d(H) = \binom{t(H)}{2} \text{ and } t(H) = \mathrm{edim}(H) - 1.$$

This follows from part d) of corollary 3.1 where p and q have to be exchanged, if necessary. The second formula generalizes $[K_2]$, Thm. 2.2 where it was assumed that p and q are coprime.

4.2 Example. For $p \in H \setminus \{0\}$ let $\mathrm{Ap}(H,p) := \{h \in H \setminus \{0\} | h \notin p + H\}$.

a) Assume that H has maximal embedding dimension p (the multiplicity of H). Then $E = \{p\} \cup \mathrm{Ap}(p,H)$ and the conditions of Corollary 3.1d) are satisfied, hence

$$d(H) = \binom{p-1}{2} \text{ and } t(H) = p - 1.$$

In particular this holds for Arf semigroups.

b) If

$$E = \{p, q\} \cup (\mathrm{Ap}(H, p) \cap \mathrm{Ap}(H, q)) = \{p, q, h_1, \ldots, h_{e-2}\},$$

then the assertions of the Corollary 3.1a) follow. In fact, since $h_i + h_j$ does not belong to E, it is non of the elements h with $h \notin p + H, h \notin q + H$, hence $h_i + h_j \in (p + H) \cup (q + H)$ as required in part a) of the corollary.

In the remainder of this section we assume that p and q are coprime, $3 \leq p < q$, and we refer to the geometrical illustration of numerical semigroups containing p and q discussed in [KW]. In this situation the relation of $\mathrm{Ap}(H, p)$ and $\mathrm{Ap}(H, q)$ to the lattice path of H is explained in the introduction of [KW]. Let $\gamma : \mathbb{Z}^2 \to \mathbb{Z}$ be the map with $\gamma(a, b) = pq - (a+1)p - (b+1)q$. The elements of $\mathrm{Ap}(H, p) \cap \mathrm{Ap}(H, q)$ correspond by γ to the "corners" of the lattice path in the sense of $[K_2]$, Remark 1.2. Therefore this remark is generalized by Corollary 3.1a) with a ring-theoretic proof to integers p, q which need not be coprime.

The next example improves $[K_2]$, Theorem 2.1.

4.3 Example. Let k be an integer with $k \geq q$ and assume that H is obtained from $< p, q >$ by closing all gaps $\geq 2k - p$ and some gaps which are $\geq k$. Then

$$\mathrm{edim}(H) - 2 \leq t(H) \leq \mathrm{edim}(H) - 1.$$

It is not hard to show that in this situation Corollary 3.1b) can be applied.

We now consider certain numerical semigroups defined by lines. For $r, s \in \mathbb{R}$ with $0 < s < p - \frac{p}{q} - 1, 0 < r < q - \frac{q}{p} - 1$ and

$$1 \leq \frac{s}{r} \leq p - s - 1$$

let $H(r, s)$ be the semigroup defined by the line $g(r, s) : \frac{x}{r} + \frac{y}{s} = 1$ in the sense of [KW], Section 3, see in particular Proposition 3.2. $H(r, s)$ is generated by p, q and $\gamma(P_i)$ with $P_i := (i, \lambda_i)$ where $\lambda_i := \lfloor s - \frac{s}{r} i \rfloor$ $(i = 0, \ldots, \lfloor r \rfloor)$, the corners of $H(r, s)$. Write $h_i := \gamma(P_i)$ $(i = 0, \ldots, \lfloor r \rfloor)$.

Then

(1) $\qquad \lambda_0 > \lambda_1 > \cdots > \lambda_{\lfloor r \rfloor} \geq 0, h_0 < h_1 < \cdots < h_{\lfloor r \rfloor}, h_i \equiv -(i+1)p \bmod q$

and

(2)

$H(r, s) \setminus \{0\}$ is the disjoint union of $\{h_0, \ldots, h_{\lfloor r \rfloor}\}$ and $p + H(r, s) \cup q + H(r, s)$.

Let $L(r, s)$ be the set of all $(x, y) \in \mathbb{N}^2$ below or on the line $g(r, s)$. For any $(i, y) \in L(r, s)$ the number $\gamma(i, y)$ has the form $\gamma(i, y) = h_i + \lambda q$ with some $\lambda \in \mathbb{N}$.

4.4 Lemma. If h_k does not belong to the minimal system E of generators of $H(r,s)$, then there exist $i,j \in \{0,\ldots,\lfloor r\rfloor\}$ such that $h_k = h_i + h_j$. For any such presentation of h_k we have $i+j+1 = k$.

Proof. By assumption there exist $h, h' \in H(r,s)$ such that $h_k = h + h', 0 < h \leq h'$. Then $h, h' \notin p + H(r,s) \cup q + H(r,s)$, since otherwise $h_k \in p + H(r,s) \cup q + H(r,s)$ contradicting (2). From (2) it follows that $h = h_i, h' = h_j$ with certain $i,j \in \{0,\ldots,\lfloor r\rfloor\},\ i \leq j$.

From $h_k = h_i + h_j,\ h_i \leq h_j$ we conclude that $h_i \leq h_j < h_k$, hence $i \leq j < k < q$, in particular $k - q < i+j+1 < k+q$. Further $h_k = h_i + h_j$ and (2) implies that $i+j+1 \equiv k \bmod q$, hence $i+j+1 = k$. $\square$

The following example generalizes $[K_2]$, Theorem 3.2 where only lines with slope -1 were considered.

4.5 Example. If $1 \leq \frac{s}{r} \leq p - s - 2$, then

$$\operatorname{edim}(H(r,s)) = \lfloor r\rfloor + 3,$$

$$d(H(r,s)) = \binom{\lfloor r\rfloor + 2}{2}$$

and

$$t(H(r,s)) = \lfloor r\rfloor + 2.$$

We first show that $E = \{p, q, h_0, \ldots, h_{\lfloor r\rfloor}\}$. Since $\lfloor s\rfloor = \lambda_0 \leq p-3$ we obtain

$$h_0 = pq - p - (\lfloor s\rfloor + 1)q \geq 2q - p > q > p$$

and thus by (1) p, q and h_0 certainly belong to E.

Next we show that

$$h_i + h_j \in q + H(r,s) \ (i,j \in \{0,\ldots,\lfloor r\rfloor\}). \tag{3}$$

Then by (2) and the lemma it follows that $E = \{p, q, h_0, \ldots, h_{\lfloor r\rfloor}\}$. We distinguish different cases.

a) If $i+j+1 \geq q$ or $\lambda_i + \lambda_j + 1 \geq p$, then $h_i + h_j = \gamma(x,y) = h_x + \lambda q$ with $(x,y) \in L(r,s)$ and $\lambda \in \mathbb{N}$ ([KW],2.3). If $i+j+1 > r$, then by the lemma $h_i + h_j \neq h_x$, hence $\lambda \geq 1$ and therefore $h_i + h_j \in q + H(r,s)$. If $i+j+1 \leq r$, hence $\lambda_i + \lambda_j + 1 \geq p$, then $h_i + h_j = \gamma(i+j+1, \lambda_i + \lambda_j + 1 - p)$.

The assumption $\frac{s}{r} \leq p - s - 2$ implies $rs + s \leq (p-2)r$, consequently

$$xs + (y+1)r = (i+j+1)s + (\lambda_i + \lambda_j - (p-2))r$$

$$\leq (i+j+1)s + 2rs - (p-2)r - s(i+j) \leq s + 2rs - (rs + s) = rs.$$

This shows that $(x, y+1) \in L(r,s)$ and $h_i + h_j = \gamma(x,y) = q + \gamma(x, y+1) \in q + H(r,s)$.

b) If $i+j+1 < q$ and $\lambda_i + \lambda_j + 1 < p$, then

$$h_i + h_j = p(q - (i+j+2)) + q((p-2) - (\lambda_i + \lambda_j)) \in <p, q> .$$

In case $p-2 > \lambda_i + \lambda_j$ we have $h_i + h_j \in q + H(r,s)$. Otherwise $h_i + h_j = pq - (i+j+2)p$ and by assumption

$$2s - \frac{s}{r}(r-1) = s + \frac{s}{r} \le p - 2 = \lambda_i + \lambda_j \le 2s - \frac{s}{r}(i+j),$$

hence $i+j+1 \le r$. Then $(i+j+1,0) \in L(r,s)$ and $h_i + h_j - q = pq - ((i+j+1)+1)p - q = \gamma(i+j+1,0) \in H(r,s)$, i. e. $h_i + h_j \in q + H(r,s)$.

We have proved in particular that $\operatorname{edim}(H(r,s)) = \lfloor r \rfloor + 3$. For the statements about the deviation and the type we show that the assumptions of Corollary 3.1d) are satisfied. We have already seen in (3) that $h_i + h_j \in q + H(r,s)$ for $i,j \in \{0,\dots,\lfloor r \rfloor\}$. Further $h_0 + p = (p - \lfloor s \rfloor - 1)q \in q + H(r,s)$ since $p - \lfloor s \rfloor - 1 \ge 1$. For $i = 1,\dots,\lfloor r \rfloor$ we have $h_i + p = h_{i-1} + (\lambda_{i-1} - \lambda_i)q$, hence also $h_i + p \in q + H(r,s)$.

Assertion c) of Corollary 3.1 can be applied to the following class of numerical semigroups. Let $p,q \in \mathbb{N}$ with $3 \le p < q$ be coprime and let $\mathbf{R}(p,q)$ be the set of all numerical semigroups H with $< p,q > \subset H \subset < p,q,r >$, where

$$r := \begin{cases} \frac{p}{2} & p \text{ even} \\ \frac{q}{2} & q \text{ even} \\ \frac{p+q}{2} & p \text{ and } q \text{ odd} \end{cases}$$

In the notation of Rosales and García-Sánchez ([RG], Chap.5) $< p,q,r >$ is the semigroup $\frac{<p,q>}{2}$.

We want to determine $d(H)$ and $t(H)$ for the semigroups in $\mathbf{R}(p,q)$. According to their geometric representation in [KW], Section 2 the $H \in \mathbf{R}(p,q)$ are in one-to-one correspondence to the lattice paths in the rectangle $\mathbf{R} \subset \mathbb{R}^2$ with the corners $(0,0),(0,p'-1),(q'-1,p'-1),(q'-1,0)$, where $p' := \lfloor \frac{p}{2} \rfloor, q' := \lfloor \frac{q}{2} \rfloor$. The only $H \in \mathbf{R}(p,q)$ with $\operatorname{edim}(H) = 2$ are $< p,q >$, corresponding to the empty lattice path, further $< \frac{p}{2}, q >$, if p is even, and $< p, \frac{q}{2} >$, if q is even. For the other $H \in \mathbf{R}(p,q)$ the elements p and q belong to the minimal system of generators E of H. It is of the form

$$E = \{p,q,h_1,\dots,h_{e-2}\}, e := \operatorname{edim}(H) \ge 3,$$

where $h_i = \gamma(a_i,b_i) = pq - (a_i+1)p - (b_i+1)q$, if (a_i,b_i) $(i = 1,\dots,e-2)$ are the corners of the lattice path defining H. ([KW], 2.10).

4.6 Example. For the non-symmetric $H \in \mathbf{R}(p,q)$ with $\operatorname{edim}(H) = 3$ and all $H \in \mathbf{R}(p,q)$ with $\operatorname{edim}(H) \ge 4$ we have

$$d(H) = \binom{t(H)}{2} \text{ and } t(H) = \operatorname{edim}(H) - 1.$$

For embedding dimension 3 this is well known. Assume therefore that $\operatorname{edim}(H) \ge 4$. We shall show that H satisfies the conditions in c) of Corollary 3.1, maybe with exchanged roles of p and q.

Since $< p,q > \subset H \subset < p,q,r >$ any h_i is of the form

$$h_i = r + \lambda_i p + \mu_i q \ (\lambda_i, \mu_i) \in \mathbb{N}^2,$$

and since

$$2r = \begin{cases} p & p \text{ even} \\ q & q \text{ even} \\ p+q & p \text{ and } q \text{ odd} \end{cases}$$

it follows that $h_i + h_j \in p + H$, if q is odd, $h_i + h_j \in q + H$, if p is odd. Therefore the first condition of part c) of corollary 3.1 is satisfied, maybe with exchanged roles of p and q.

Assume that the lattice path defining H starts at $(0, b-1)$ and ends at $(a-1, 0)$. We show that for odd q the element $h := p + \gamma(0,b) = (p-b-1)q$ and for odd p the element $h := q + \gamma(a,0) = (q-a-1)p$ satisfies the remaining conditions of Corollary 3.1c).

For odd q we have to show
(4) $h \in H \setminus \{0\}, h \notin p + H$ and $h \notin \gamma(a_i, b_i) + H$ $(i = 1, \ldots, e-2)$
and
(5) $h + q \in p + H$ and $h + \gamma(a_i, b_i) \in p + H$ $(i = 1, \ldots, e-2)$.

As for (4): We have $h = (p-b-1)q \in H \setminus \{0\}$ since $p - 1 > p' \geq b$, further $h - p = \gamma(0,b) \notin H$. Also $h - \gamma(a_i, b_i) = \gamma(q - 2 - a_i, b - b_i - 1) \notin H$, as $q - 2 - a_i \geq q - 2 - (\frac{q-1}{2} - 1) = \frac{q-1}{2} = q' > q' - 1$ and $b - 1 - b_i \geq 0$, hence $(q - 2 - a_i, b - b_i - 1) \in \mathbb{N}^2$ is a point outside of the rectangle $\mathbf{R}$.

As for (5): We have $h + q = p + \gamma(0, b-1) \in p + H$, since $(0, b-1)$ is on the lattice path of H. If $b + b_i < p - 1$, then $h + \gamma(a_i, b_i) = (q-1-a_i)p + (p-2-(b+b_i))q \in p + < p, q >$.

If $b + b_i \geq p - 1$, then $p' - 1 \geq b_i \geq p - 1 - b \geq p - 1 - p'$. This can only happen, if p is even and $b_i = \frac{p}{2} - 1 = b - 1$. Then $(a_i, b_i) = (a_1, b-1)$ is the first corner of H. Since $\mathrm{edim}(H) \geq 4$ there is also a second one, hence $a_1 + 1 \leq a - 1$. It follows that $h + \gamma(a_1, b_1) = p + \gamma(0, b) + \gamma(a_1, p' - 1) = p + \gamma(a_1 + 1, 0) \in p + H$.

For odd p we can argue in the same manner exchanging p and q.

In the appendix for the $H \in \mathbf{R}(p,q)$ as in Example 4.6 a minimal system of generators of the relation ideal of $K[[H]]$ in terms of p, q and the coordinates of the corners of H will be given.

Appendix: The relation ideal of the $H \in \mathbf{R}(p,q)$

For $H \in \mathbf{R}(p,q)$ with $\mathrm{edim}(H) \geq 3$ let (a_i, b_i) $(i = 1, \ldots, s := \mathrm{edim}(H) - 2)$ be the corners of H and I be the relation ideal of $K[[H]]$ in $K[[X, Y, X_1, \ldots, X_s]]$ for $X \mapsto t^p, Y \mapsto t^q, X_i \mapsto t^{\gamma(a_i, b_i)}$. We know that $\{p, q, \gamma(a_i, b_i) \ (i = 1, \ldots, s)\}$ is the minimal system of generators of H.

Theorem. For the non-symmetric $H \in \mathbf{R}(p,q)$ with $\mathrm{edim}(H) = 3$ and all $H \in \mathbf{R}(p,q)$ with $\mathrm{edim}(H) \geq 4$ the ideal I has the following minimal system of generators

$$G := \{X_iX_j - X^{q-a_i-a_j-2}Y^{p-b_i-b_j-2}\}_{i,j=1,\dots,s,i\leq j}$$

$$\cup\{Y^{b_i-b_{i+1}}X_i - X^{a_{i+1}-a_i}X_{i+1}\}_{i=1,\dots,s-1}$$

$$\cup\{Y^{p-b_1-1} - X^{a_1+1}X_1, Y^{b_s+1}X_s - X^{q-a_s-1}\}.$$

Proof. Note that $q - a_i - a_j - 2 \in \mathbb{N}$ and $p - b_i - b_j - 2 \in \mathbb{N}$ since $a_i, a_j < q', b_i, b_j < p'$. Further $q - a_i - a_j - 2 = 0$ if and only if q is even, $i = j = s$ and $a_s = \frac{q}{2} - 1$ (resp. $p - b_i - b_j - 2 = 0$ if and only if p is even, $i = j = 1$ and $b_1 = \frac{p}{2} - 1$). In both cases H is symmetric with $\mathrm{edim}(H) = 3$ ([KW], 2.8) which was excluded. We assume that X occurs in all polynomials of G, else we have to exchange the roles of X and Y in the following.

Using the lattice path of H and the function γ we find the relations

$$\gamma(a_i,b_i)+\gamma(a_j,b_j) = (q-a_i-a_j-2)p+(p-b_i-b_j-2)q \ (i,j=1,\dots,s, i\leq j)$$

$$(b_i - b_{i+1})q + \gamma(a_i,b_i) = (a_{i+1}-a_i)p + \gamma(a_{i+1},b_{i+1}) \ (i = 1,\dots,s-1)$$

$$(p-b_1-1)q = (a_1+1)p + \gamma(a_1,b_1) \text{ and } (b_s+1)q + \gamma(a_s,b_s) = (q-a_s-1)p$$

which correspond to the polynomials of G. Thus these polynomials belong to I. Let J be the ideal they generate.

To show that G is a minimal system of generators we go over to $K[[H]]/(t^p) = K[[Y, X_1,\dots,X_s]]/\bar{I}$ where $\bar{I}$ is the image of I by the map $X \mapsto 0$.

Let $\bar{J} \subset K[[Y, X_1,\dots,X_s]]$ be the ideal generated by the images

$$X_iX_j \ (i,j = 1,\dots,s, i\leq j)$$

$$Y^{b_i-b_{i+1}}X_i \ (i=1,\dots,s-1)$$

$$Y^{p-b_1-1}\text{and } Y^{b_s+1}X_s.$$

of the polynomials from G. These monomials clearly form a minimal system of generators of $\bar{J}$ unless $s = 1, p$ is even and $b_1 = \frac{p}{2} - 1$. But this case was excluded. It follows that G is a minimal system of generators of J.

It remains to be shown that $I = J$. We have

$$K[[Y, X_1,\dots,X_s]]/\bar{J} \cong$$

$$K[Y]/(Y^{p-b_1-1}) \oplus \bigoplus_{i=1}^{s-1} K[Y]X_i/(Y^{b_i-b_{i+1}}X_i) \oplus K[Y]X_s/(Y^{b_s+1}X_s),$$

hence

$$\dim_K(K[[Y,X_1,\ldots,X_s]]\bar{J}) = p - b_1 - 1 + \sum_{i=1}^{s-1}(b_i - b_{i+1}) + b_s + 1 = p$$

$$= \dim_K(K[[H]]/(t^p)) = \dim_K(K[[Y,X_1,\ldots,X_s]]\bar{I}).$$

It follows that $\bar{I} = \bar{J}$, i.e. $I + (X) = J + (X)$, and by Nakayama $I = J$. $\square$

Since $|G| = \binom{s+1}{2} + s + 1 = \binom{\mathrm{edim}(R)-1}{2} + \mathrm{edim}(R) - 1$ we have $d(H) = \binom{\mathrm{edim}(R)-1}{2}$ for which in 4.6 a different proof was given.

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