Studying the catenary and the tame degrees in 4-generated symmetric non complete intersection numerical semigroups

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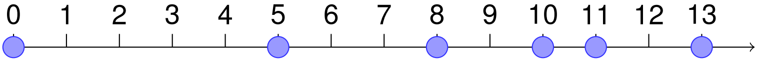
Example



$S = \langle 5, 8, 11, 14, 17 \rangle$
$\mathrm{e}(S) = 5$

Setup

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- The factorization set of $s \in S$ is the set of the solutions to $x_1 n_1 + \cdots + x_p n_p = s$, $\mathrm{Z}(s) = \{x \in \mathbb{N}^e \mid \varphi(x) = s\} = \varphi^{-1}(s)$.

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- The length of $x \in Z(s)$ is $|x| = x_1 + \cdots + x_p$.
- Given another factorization $y = (y_1, \ldots, y_p)$, the distance between x and y is
  $\mathrm{d}(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\}$,
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- A presentation of S is a congruence σ on $\mathbb{N}^p$ contained in $\ker \varphi$.

The graph G_n

Let $S = \langle n_1, \ldots, n_p \rangle$ be a p-generated numerical semigroup, $n \in S$ we define the graph $G_n = (V_n, E_n)$ such that, for any $1 \leq i, j \leq p$, $i \neq j$:

- $n_i \in V_n \Leftrightarrow n - n_i \in S$;
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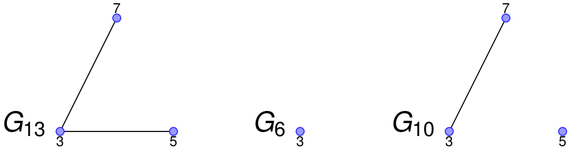

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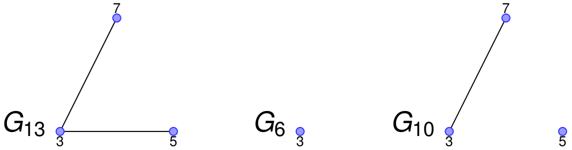

We define

$$\text{Betti}(S) = \{n \in S \mid G_n \text{ is not connected}\}.$$

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- pick $\alpha_i \in C_i$;
- set $\sigma_n = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), \ldots, (\alpha_1, \alpha_t)\}$.

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Actually,

$$\sigma = \bigcup_{b \in \mathrm{Betti}(S)} \sigma_b.$$

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S is a complete intersection $\Rightarrow$ S symmetric.

If $e(S) \leq 3$, S is a complete intersection $\Leftrightarrow$ S symmetric (Herzog).

The catenary degree

The catenary degree of $s \in S$, $\mathrm{c}(s)$, is the minimum nonnegative integer N such that for any two factorizations x and y of s, there exists a sequence of factorizations $x_1, \ldots, x_t$ of s such that

- $x_1 = x$, $x_t = y$,
- for all $i \in \{1, \ldots, t-1\}$, $\mathrm{d}(x_i, x_{i+1}) \leq N$.

The catenary degree of S, $\mathrm{c}(S)$, is the supremum (maximum) of the catenary degrees of the elements of S.

Example: $66 \in S = \langle 6, 9, 11 \rangle$, $\mathsf{c}(66) = 4$

The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$\mathsf{Z}(66) = \{(0,0,6), (1,3,3), (2,6,0), (4,1,3), (5,4,0), (8,2,0), (11,0,0)\}$$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is 11.

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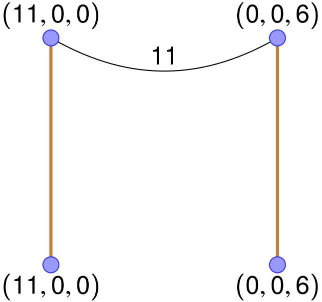

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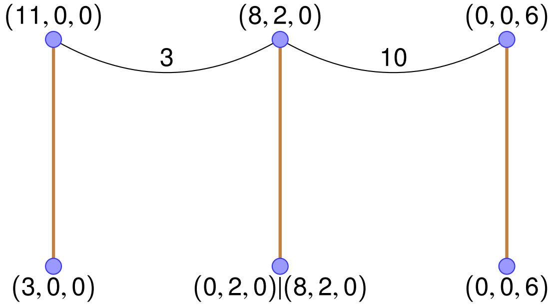

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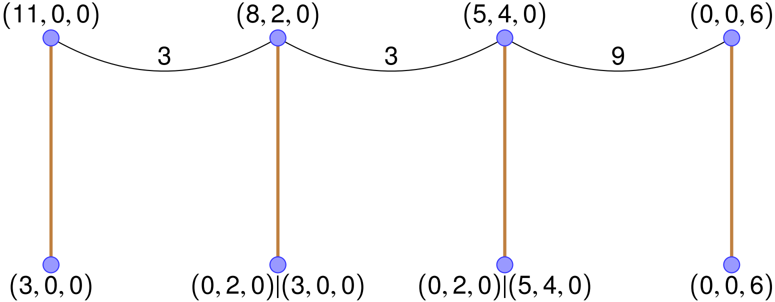

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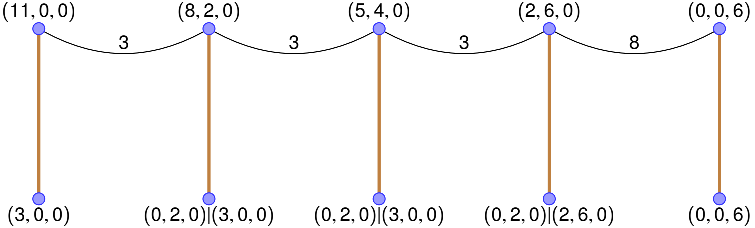

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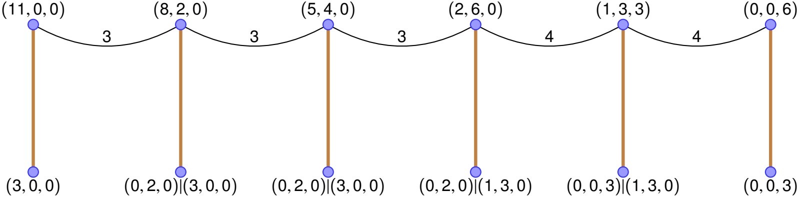

The tame degree

The tame degree of S, $\mathrm{t}(S)$, is defined as the minimum N such that for any $s \in S$ and any factorization x of s, if $s - n_i \in S$ for some $i \in \{1, \ldots, p\}$, then there exists another factorization y of s such that $\mathrm{d}(x, y) \leq N$ and the ith coordinate of y is nonzero (n_i "occurs" in this factorization).

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Goal: Say if the inequality is strict or not for numerical semigroups S with $e(S) = 4$ that are symmetric but not complete intersection.

Bresinsky's theorem

The numerical semigroup S is 4-generated symmetric, not complete intersection, if and only if there are integers α_i, $1 \leq i \leq 4$, α_{ij}, $i, j \in \{21, 31, 32, 42, 13, 43, 14, 24\}$, s.t.:

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- $n_1 = \alpha_2 \alpha_3 \alpha_{14} + \alpha_{32} \alpha_{13} \alpha_{24}$, $n_2 = \alpha_3 \alpha_4 \alpha_{21} + \alpha_{31} \alpha_{43} \alpha_{24}$, $n_3 = \alpha_1 \alpha_4 \alpha_{32} + \alpha_{14} \alpha_{42} \alpha_{31}$, $n_4 = \alpha_1 \alpha_2 \alpha_{43} + \alpha_{42} \alpha_{21} \alpha_{13}$.

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Then

$$\mathrm{Betti}(S) = \begin{cases} b_1 = \alpha_1\,n_1 = \alpha_{13}\,n_3 + \alpha_{14}\,n_4 \\ b_2 = \alpha_2\,n_2 = \alpha_{21}\,n_1 + \alpha_{24}\,n_4 \\ b_3 = \alpha_3\,n_3 = \alpha_{31}\,n_1 + \alpha_{32}\,n_2 \\ b_4 = \alpha_4\,n_4 = \alpha_{42}\,n_2 + \alpha_{43}\,n_3 \\ b_5 = \alpha_{21}\,n_1 + \alpha_{43}\,n_3 = \alpha_{32}\,n_2 + \alpha_{14}\,n_4 \end{cases}$$

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- The catenary degree is reached in one of the Betti elements, $c(S) = \max\{c(b) \mid b \in \mathrm{Betti}(S)\}$;

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Then,

$$c(S) = \max\{\alpha_1, \alpha_{13} + \alpha_{14}, \alpha_2, \alpha_{21} + \alpha_{24}, \alpha_3, \alpha_{31} + \alpha_{32}, \alpha_4, \alpha_{42} + \alpha_{43}, \alpha_{21} + \alpha_{43}, \alpha_{32} + \alpha_{14}\}.$$

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$\mathrm{NC}(S) = \{n \in S \mid G_n \text{ is not complete}\}$
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Known: $t(S) = \max\{t(n) \mid n \in \mathrm{Prim}(S) \cap \mathrm{NC}(S)\}$, where

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$$x_1 n_1 + x_2 n_2 + x_3 n_3 + x_4 n_4 - y_1 n_1 - y_2 n_2 - y_3 n_3 - y_4 n_4 = 0$$

and $x \neq y\}$

$\mathrm{NC}(S) = \{n \in S \mid G_n$ is not complete$\}$
$\mathrm{Betti}(S) \subseteq \mathrm{Prim}(S) \cap \mathrm{NC}(S)$. But since each Betti element b_i has just two factorizations with gcd $= (0, 0, 0, 0)$, $t(b_i) = c(b_i)$

Idea: find an element n in $(\mathrm{Prim}(S) \cap \mathrm{NC}(S)) \setminus \mathrm{Betti}(S)$ s.t. $t(n) > c(S)$.

The case $c(S) = \alpha_i$, $i \in \{1, 2, 3, 4\}$

Take $k = \min\{h \in \mathbb{N} \mid hn_i - n_j \in S, j \equiv i + 1 \pmod 4\}$.
$(k > \alpha_i)$

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$$t(kn_i) \geq d(z, z') \geq k > \alpha_i = c(S)$$

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$$t(kn_i) \geq d(z, z') \geq k > \alpha_i = c(S)$$

$$\neg \Downarrow$$

$$t(S) \geq t(kn_i) > c(S)$$

Thank you