On proportionally modular affine semigroups

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joint work with J.I. García-García and M.A. Moreno-Frías

Definition (Rosales, García-Sánchez, García-García, Urbano-Blanco 2003)

Proportionally modular numerical semigroups is the set of the non-negative integer solutions of the inequality

$$ax \mod b \leq cx$$

with a, b and c positive integers.

Remark

Proportionally modular numerical semigroups...

- ... are generated by closed intervals, $\mathbb{N} \cap (\cup_{i \in \mathbb{N}} i[\alpha, \beta])$.
- ... have minimal generating sets related with Bézout sequences.

Definition

Proportionally modular affine semigroup, the set

$$\{x \in \mathbb{N}^p | f(x) \mod b \leq g(x)\},$$

where f and g are nonnull linear functions $f, g : \mathbb{Q}^p \to \mathbb{Q}$, and $b \in \mathbb{N}$.

Problem

FIRST PROBLEM...

... TO COMPUTE ITS MINIMAL SYSTEM OF GENERATORS!

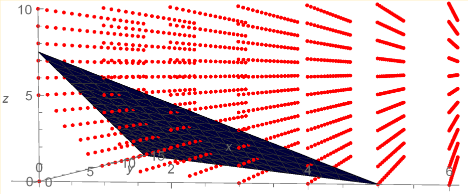
Sketch of the algorithm

Input: $f(x) \mod b \leq g(x) = g_1x_1 + \cdots + g_px_p$.
Output: A generating set of S.

1: **if** $g_1, \ldots, g_p < 0$ **then return** $\{\mathbf{0}\}$.

2: **if** $g_1, \ldots, g_p > 0$ **then** a generating set is obtained from $\mathbb{N}^p \setminus S$.
return generating set.

$$5x + 2y + 3z \mod 15 \leq 3x + y + 2z$$



Sketch of the algorithm

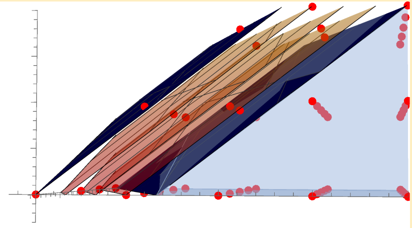
Input: $f(x) \mod b \leq g(x) = g_1x_1 + \cdots + g_px_p$.
Output: A generating set of S.

1: **if** $g_ig_j \leq 0$ **then**

2: Compute M_{dk} the minimal $\mathbb{N}-$solutions of

$$\begin{cases} f(x) \mod b = k, \\ g(x) = d. \end{cases}$$

with $k = 0, \ldots, d$ and $d = 1, \ldots, b-1$.



Sketch of the algorithm

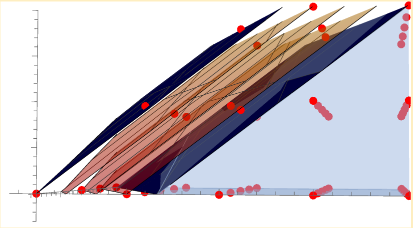
Input: $f(x) \mod b \leq g(x) = g_1x_1 + \cdots + g_px_p$.
Output: A generating set of S.

1: **if** $g_ig_j \leq 0$ **then**
2: Compute M_{dk}
3: Compute $\widetilde{C}$ the minimal generating set of

$$\{x \in \mathbb{N}^p \mid g(x) \geq b\}$$

$$\cup$$

$$(S \cap \{x \in \mathbb{N}^p \mid g(x) = 0\})$$



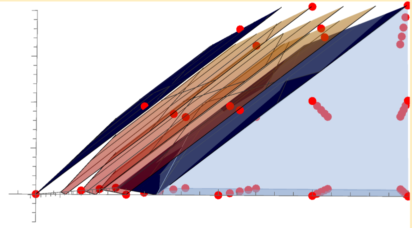
Sketch of the algorithm

Input: $f(x) \mod b \leq g(x) = g_1 x_1 + \cdots + g_p x_p$.
Output: A generating set of S.

1: **if** $g_i g_j \leq 0$ **then**
2: Compute M_{dk}
3: Compute $\widetilde{C}$
4: A generating set of S is

$$\widetilde{C} \cup (\cup_{d=1}^{b-1} \cup_{k=0}^{d} M_{dk})$$

$5x + 2y + z \mod 4 \leq 3x + y - 4z$



Notation

Proportionally modular Diophantine inequality into two variables:

$$f(x,y) \mod b \leq g(x,y)$$

where $f(x,y) = f_1x + f_2y$, $g(x,y) = g_1x + g_2y$ with $b \in \mathbb{N}$ and $f_1, f_2, g_1, g_2 \in \mathbb{Z}$.

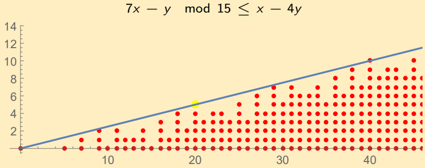
Definition

Assume $g(x,y) = g_1x + g_2y$ with $g_1g_2 \leq 0$. Denote by $\mathfrak{u}$ the generator of the semigroup given by the $\mathbb{N}$-solutions of

$$\begin{cases} g_1x + g_2y = 0, \\ f_1x + f_2y \mod b = 0. \end{cases}$$

Lemma

Assume $g(x,y) = g_1x + g_2y$ with $g_1g_2 \leq 0$, let $v, w \in \mathbb{N}^2$ such that $v + \mathfrak{u} = w$. Then, $v \in S$ if and only if $w \in S$.

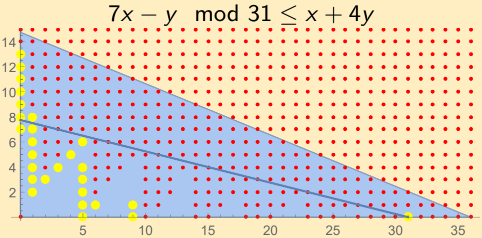


Algorithm

Input: $f(x) \mod b \leq g(x) = g_1 x + g_2 y$.
Output: The minimal generating set of S.

1: **if** $g_1, g_2 < 0$ **then return** $\{(0,0)\}$.

2: **if** $g_1, g_2 > 0$ **then** the minimal generating set H of S is obtained from $\mathbb{N}^2 \setminus S$. **return** H.



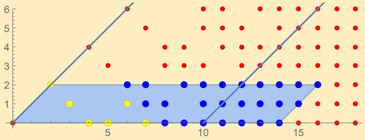

```
In[1]:= ProporcionallyModularAffineSemigroupN2[7,-1,31,1,4]
Out[1]= {{0.,7.},{0.,8.},{0.,9.},{0.,10.},{0.,11.},{0.,12.},{0.,13.},{1.,2.},{1.,3.},
{1.,4.},{1.,5.},{1.,6.},{1.,7.},{1.,8.},{2.,3.},{3.,4.},{4.,5.},{5.,0.},{5.,1.},
{5.,2.},{5.,3.},{5.,4.},{5.,6.},{6.,1.},{9.,0.},{9.,1.},{31.,0.}}
```

Algorithm

Input: $f(x) \mod b \leq g(x) = g_1 x + g_2 y$.
Output: The minimal generating set of S.

1: **if** $g_1 g_2 \leq 0$ **then**
2: Compute the vector $\mathfrak{u}$.
3: **if** $g_1 \geq 0$ **then** $\widetilde{S} := \{(x,0) \mid f(x,0) \mod b \leq g(x,0)\}$.
4: **if** $g_1 < 0$ **then** $\widetilde{S} := \{(0,y) \mid f(0,y) \mod b \leq g(0,y)\}$.
5: Compute the minimum minimal generator $\widetilde{\mathfrak{u}}$ of $\widetilde{S}$.
6: $w := \{x \in \mathbb{R}^2_+ | g(x) = b\} \cap (OX \cup OY)$.
7: $\mathcal{G} := S \cap \text{ConvexHull}(\{O, \mathfrak{u}, \mathfrak{u} + w + \widetilde{\mathfrak{u}}, w + \widetilde{\mathfrak{u}}\})$.
8: Obtain H a minimal system of generators from $\mathcal{G}$. **return** H.

$3x + 2y \mod 10 \leq x - y$



```
In[2]:= ProporcionallyModularAffineSemigroupN2[3, 2, 10, 1, -1]
Out[2]= {{2.,2.},{3.,1.},{4.,0.},{5.,0.},{6.,1.},{7.,0.}}
```

Definition

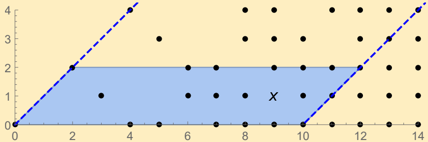
Given an affine semigroup T, $q \notin T$ is a Frobenius vector if q in the group $G(T)$ such that $(q + \mathrm{Tint}(\mathrm{L}(T))) \cap G(T) \subset S \setminus \{0\}$. A Frobenius vector is called minimal Frobenius vector if it is minimal with respect to the product ordering on $\mathbb{N}^p$.

Proposition

Let $S \subset \mathbb{N}^2$ be a nontrivial proportionally modular semigroup:

- If $g_1 g_2 \leq 0$, the unique minimal Frobenius vector is the minimal integer element in ConvexHull$(\{O, u, w, w+u\}) \setminus S$ closest to $\{x \in \mathbb{R}^2 | g(x) = b\}$.

$$3x + 2y \mod 10 \leq x - y$$

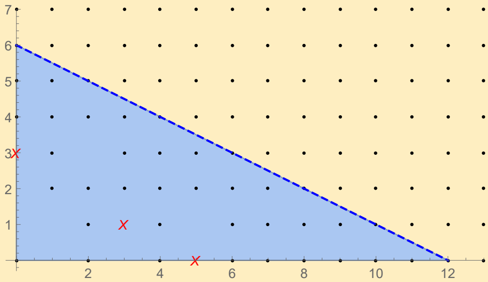


Proposition

Let $S \subset \mathbb{N}^2$ be a nontrivial proportionally modular semigroup:

- If $g_1 g_2 > 0$, a minimal Frobenius vector is a maximal element in $(\text{ConvexHull}(\{O, w_1, w_2\}) \cap \mathbb{N}^2) \setminus S$, or an element ω_1 in $(\text{ConvexHull}(\{O, w_1, w_2\}) \cap \mathbb{N}^2) \setminus S$ such that there is no maximal element belonging to $(\text{ConvexHull}(\{O, w_1, w_2\}) \cap \mathbb{N}^2) \setminus S$ in $\omega_1 + \text{Tint}(\text{L}(S))$.

$$7x - y \mod 31 \leq x + 4y$$



Proposition (García-García, Vigneron-Tenorio 2014)

Let $T \subseteq \mathbb{N}^2$ be an affine simplicial semigroup, the following conditions are equivalent:

1. T is Cohen-Macaulay.
2. For all $v \in (\mathrm{L}(T) \cap \mathbb{N}^2) \setminus T$, $v + s_1$ or $v + s_2$ does not belong to T where s_1 and s_2 are minimal generators of T such that $\mathrm{L}(T) = \langle s_1, s_2 \rangle$.

Corollary

Every proportionally modular semigroup with $g_1 g_2 \leq 0$ is Cohen-Macaulay.

Theorem (Rosales, García-Sánchez 1998)

For a given affine simplicial semigroup T, the following conditions are equivalent:

1. *T is Gorenstein.*
2. *T is Cohen-Macaulay and $\cap_{i=1}^{2}\mathrm{Ap}(s_i)$ has a unique maximal element (respect to the order defined by T) where s_1 and s_2 are minimal generators of T such that $\mathrm{L}(T)=\langle s_1,s_2\rangle$.*

$\mathfrak{u}$, $\widetilde{\mathfrak{u}}$ minimal generators of S with $\mathrm{L}(S)=\langle \mathfrak{u},\widetilde{\mathfrak{u}}\rangle$.

Lemma

Let S be a proportional modular semigroup with $g_1g_2\leq 0$. The set $\mathrm{Ap}(\mathfrak{u})\cap\mathrm{Ap}(\widetilde{\mathfrak{u}})=\{h\in\mathcal{G}|h-\mathfrak{u},h-\widetilde{\mathfrak{u}}\notin S\}$.

Corollary

Let S be a proportional modular semigroup with $g_1g_2\leq 0$. The semigroup S is Gorenstein iff there exists a unique maximal element in $\{h\in\mathcal{G}|h-\mathfrak{u},h-\widetilde{\mathfrak{u}}\notin S\}$.

Theorem (García-Sánchez, Rosales 2002)

The following conditions are equivalent:

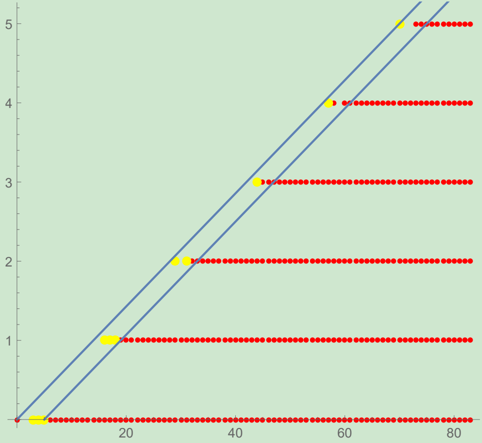
1. T *is an affine Buchsbaum simplicial semigroup.*
2. $\overline{T} = \{s \in \mathbb{N}^p | s + s_i \in T, \forall i = 1, \ldots, t\}$ *is Cohen-Macaulay.*

Corollary

Let $S \subset \mathbb{N}^2$ *be a proportional modular semigroup with* $g_1 g_2 \leq 0$. *Then,* S *is Buchsbaum.*

Example (Cohen-Macaulay, Gorenstein and Buchsbaum)

$$7x - y \mod 5 \leq x - 14y$$



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Grupo de investigación
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Thanks for your attention!