# On numerical semigroups closed with respect to the action of affine maps

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1. Thabit and Mersenne numerical semigroups

2. On affine maps and numerical semigroups
   - Submonoids of $(\mathbb{N}, +)$ closed w.r.t. affine maps
   - The smallest $\vartheta_{a,b}$-semigroup containing a positive integer

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# Thabit numerical semigroups

**Definition [Rosales, Branco and Torrão, 2015]**

A numerical semigroup $S$ is a Thabit numerical semigroup if there exists $n \in \mathbb{N}$ such that

$$S = \langle \{3 \cdot 2^{n+i} - 1 : i \in \mathbb{N}\} \rangle.$$

**Remark**

$S$ is a numerical semigroup since its generators are coprime.

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**Example**

Let $n = 2$.

Then $S$ is generated by

$$3 \cdot 4 - 1 = 11$$

$$3 \cdot 8 - 1 = 23$$

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**Proposition [RBT]**

Let $n$ be a non-negative integer and

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# Submonoids of $(\mathbb{N},+)$ closed w.r.t. affine maps

**Definition**

For $a \in \mathbb{N}^* := \mathbb{N}\setminus\{0\}$ and $b \in \mathbb{N}$ we define the map

$$\vartheta_{a,b} : \begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ x & \mapsto & ax+b \end{array}$$

**Definition**

A subsemigroup $G$ of $(\mathbb{N},+)$ containing 0 is a $\vartheta_{a,b}$-semigroup if $\vartheta_{a,b}(y) \in G$ for any $y \in G\setminus\{0\}$.

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# $\vartheta_{a,b}$-semigroups

## Remark

A $\vartheta_{a,b}$-semigroup is not necessarily a numerical semigroup (two examples follow).

## Example 1

$2\mathbb{N} = \{2n : n \in \mathbb{N}\}$ is a $\vartheta_{a,b}$-semigroup for any $a \in \mathbb{N}^*$ and $b \in 2\mathbb{N}$.

## Example 2

$M(b,n) := \langle \{b^{n+i} - 1 : i \in \mathbb{N}\} \rangle$, where $b \in \mathbb{N} \setminus \{0,1,2\}$ and $n$ is a positive integer, is a $\vartheta_{b,b-1}$-semigroup.
(For the details see Rosales, Branco and Torrão, 2016).

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# The smallest $\vartheta_{a,b}$-semigroup containing a positive integer

**Definition**

Let $\{a, b\} \subseteq \mathbb{N}^*$ and $c \in \mathbb{N}\backslash\{0, 1\}$ such that $\gcd(b, c) = 1$. We denote by $G_{a,b}(c)$ the smallest $\vartheta_{a,b}$-semigroup containing $c$.

**Remark**

If $a, b$ and $c$ are as above, then $G_{a,b}(c)$ is a numerical semigroup.

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Then

$$T(n) = G_{2,1}(3 \cdot 2^n - 1),$$
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# Apéry set for $G_{a,b}(c)$

**Definition**

Let $\{a, b, c\} \subseteq \mathbb{N}^*$, where $\gcd(b, c) = 1$, and

$$G := G_{a,b}(c).$$

Then $\mathrm{Ap}(G, c) := \{s \in G : s - c \notin G\}$.

**Remark**

We have that $|\mathrm{Ap}(G, c)| = c$. We can write

$$\mathrm{Ap}(G, c) = \{x_i : 0 \leq i \leq c - 1\},$$

where $x_0 = 0 < x_1 < \cdots < x_{c-1}$

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# Construction of the element $x_l$

- We write $l = \sum_i j_i \cdot s_i(a)$, where

$$s_i(a) := \begin{cases} 0 & \text{if } i = 0, \\ \sum_{k=0}^{i-1} a^k & \text{if } i > 0, \end{cases}$$

  and any $j_i \in \{0, 1, \dots, a\}$.
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# Example: $G_{3,1}(3)$

We have that

$$a = 3, \quad b = 1, \quad c = 3,$$

and $\mathrm{Ap}(G_{3,1}(3), 3) = \{x_0 = 0, x_1, x_2\}$.

$$1 = 1 \cdot s_1(3) \Rightarrow x_1 = 1 \cdot f_1(3, 1, 3) = 10,$$
$$2 = 2 \cdot s_1(3) \Rightarrow x_2 = 2 \cdot f_1(3, 1, 3) = 20.$$

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We construct the set of integers smaller than or equal to 20 belonging to $G_{3,1}(3)$:

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We have that

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$$a = 2, \quad b = 3, \quad c = 4,$$

and $\mathrm{Ap}(G_{2,3}(4), 4) = \{x_0 = 0, x_1, x_2, x_3\}$.

$$1 = 1 \cdot s_1(2) \Rightarrow x_1 = 1 \cdot t_1(2,3,4) = \mathit{11},$$
$$2 = 2 \cdot s_1(2) \Rightarrow x_2 = 2 \cdot t_1(2,3,4) = \mathit{22},$$
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We construct the set of integers smaller than or equal to 27 belonging to $G_{2,3}(4)$:

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# Properties of $G_{a,b}(c)$

The following hold for $G_{a,b}(c)$:

- $F(G_{a,b}(c)) = x_{c-1} - c$;
- $g(G_{a,b}(c)) = \frac{1}{c} \cdot \sum_{l=1}^{c-1} x_l - \frac{c-1}{2}$;
- if $\bar{k} := \min\{k \in \mathbb{N} : s_k(a) > c - 1\}$, then

$$\{t_k(a, b, c) : k \in \mathbb{N} \text{ and } 0 \leq k \leq \bar{k}\}$$

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# References

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