On the Markov complexity of numerical semigroups

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Toric ideals

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathbb{Z}^m$ be a vector configuration in $\mathbb{Q}^m$ and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_n\mathbf{a}_n \mid l_i \in \mathbb{N}_0\}$ the corresponding affine semigroup.
Let $A = [\mathbf{a}_1 \ldots \mathbf{a}_n] \in \mathbb{Z}^{m\times n}$ be an integer matrix with columns $\{\mathbf{a}_i\}$. For a vector $\mathbf{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ we let $\mathbf{u}^+$, $\mathbf{u}^-$ be the unique vectors in $\mathbb{N}^n$ with disjoint support such that $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$.

Definition

The toric ideal I_A of A is the ideal in $K[x_1, \cdots, x_n]$ generated by all binomials of the form $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ where $\mathbf{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$.

Markov basis

A Markov basis of A is a finite subset M of $\mathrm{Ker}_{\mathbb{Z}}(A)$ such that whenever $\mathbf{w}, \mathbf{u} \in \mathbb{N}^n$ and $\mathbf{w} - \mathbf{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ (i.e. $A\mathbf{w}^t = A\mathbf{u}^t$), there exists a subset $\{\mathbf{v}_i : i = 1, \ldots, s\}$ of M that connects $\mathbf{w}$ to $\mathbf{u}$. This means that $(\mathbf{w} - \sum_{i=1}^{p} \mathbf{v}_i) \in \mathbb{N}^n$ for all $1 \leq p \leq s$ and $\mathbf{w} - \mathbf{u} = \sum_{i=1}^{s} \mathbf{v}_i$. A Markov basis M of A is minimal if no subset of M is a Markov basis of A.

Theorem

(Diaconis-Sturmfels 1998) M is a minimal Markov basis of A if and only if the set $\{B(\mathbf{u}) = x^{\mathbf{u}^+} - x^{\mathbf{u}^-} : \mathbf{u} \in M\}$ is a minimal generating set of I_A.

Graver basis

Definition

An irreducible binomial $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ in I_A is called *primitive* if there exists no other binomial $x^{\mathbf{v}^+} - x^{\mathbf{v}^-} \in I_A$ such that $x^{\mathbf{v}^+}$ divides $x^{\mathbf{u}^+}$ and $x^{\mathbf{v}^-}$ divides $x^{\mathbf{u}^-}$.

Definition

The set of all primitive binomials $x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ of a toric ideal I_A is called the Graver basis of I_A. The set of all $\mathbf{u}$ such that $B(\mathbf{u}) = x^{\mathbf{u}^+} - x^{\mathbf{u}^-}$ is in the Graver basis of I_A is called the Graver basis of A.

Graver basis

The Graver basis of a toric ideal I_A is very important.

- Every circuit belongs to the Graver basis
- Every reduced Gröbner basis is a subset of the Graver basis
- The universal Gröbner basis is a subset of the Graver basis
- If the semigroup $\mathbb{N}A$ is positive ($\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^n = \{0\}$) then all minimal systems of generators (minimal Markov bases) are subsets of the Graver basis
- If the semigroup $\mathbb{N}A$ is not positive ($\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^n \neq \{0\}$) then there is atleast one minimal system of generators (minimal Markov basis) that is a subset of the Graver basis
- The Graver basis contains Markov bases for all subconfigurations of A

Lawrence liftings

For $A \in \mathbb{M}^{m\times n}(\mathbb{Z})$ and $r \geq 2$, the r–th *Lawrence lifting* of A is denoted by $A^{(r)}$ and is the $(rm+n) \times rn$ matrix

$$A^{(r)} = \left(\begin{array}{cccc} \multicolumn{4}{c}{\overbrace{}^{r-\text{times}}} \\ A & 0 & & 0 \\ 0 & A & & 0 \\ & & \ddots & \\ 0 & 0 & & A \\ I_n & I_n & \cdots & I_n \end{array} \right).$$

We identify an element of $\mathrm{Ker}_{\mathbb{Z}}(A^{(r)})$ with an $r \times n$ matrix: each row of this matrix corresponds to an element of $\mathrm{Ker}_{\mathbb{Z}}(A)$ and the sum of its rows is zero. The *type* of an element of $\mathrm{Ker}_{\mathbb{Z}}(A^{(r)})$ is the number of nonzero rows of this matrix.

Lawrence liftings

Let σ be a permutation of $\{1, 2, \cdots, r\}$, if

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \\ \mathbf{u}_r \end{pmatrix} \in \mathrm{Ker}_{\mathbb{Z}}(A^{(r)}) \text{ then } \begin{pmatrix} \mathbf{u}_{\sigma(1)} \\ \mathbf{u}_{\sigma(2)} \\ \mathbf{u}_{\sigma(3)} \\ \\ \mathbf{u}_{\sigma(r)} \end{pmatrix} \in \mathrm{Ker}_{\mathbb{Z}}(A^{(r)}).$$

The same result is true if in the position of $\mathrm{Ker}_{\mathbb{Z}}(A^{(r)}$ we put the Graver basis of $A^{(r)}$ or the universal Markov basis of $A^{(r)}$.

Definition

The universal Markov basis of $A^{(r)}$ is the union of all minimal Markov bases.

Lawrence liftings

If $\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \\ \mathbf{u}_r \end{pmatrix} \in \mathrm{Ker}_{\mathbb{Z}}(A^{(r)})$ then $\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \\ \mathbf{u}_r \\ \mathbf{0} \end{pmatrix} \in \mathrm{Ker}_{\mathbb{Z}}(A^{(r+1)})$.

The same result is true if in the position of $\mathrm{Ker}_{\mathbb{Z}}(A^{(r)})$ we put the Graver basis of $A^{(r)}$ (and $A^{(r+1)}$) or the universal Markov basis of $A^{(r)}$ (and $A^{(r+1)}$).

Markov complexity

The study of $A^{(r)}$, for $A \in M^{m \times n}(\mathbb{Z})$ was motivated by considerations of hierarchical models in Algebraic Statistics. Aoki and Takemura in 2002 while studying Markov bases for the Lawrence liftings of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

they proved that the type of any element in a Markov basis of $A^{(r)}$ is atmost 5. While the type of any element in the Graver basis of $A^{(r)}$ is atmost 9.

Markov complexity

Definition

The Markov complexity of A is the largest type of any vector in the universal Markov basis of $A^{(r)}$ as r varies.

Definition

The Graver complexity of A is the largest type of any vector in the Graver basis of $A^{(r)}$, as r varies.

In the previous example the Markov complexity is 5 and the Graver complexity is 9.

Markov complexity

Theorem

Sturmfels and Santos (2003)
The Graver complexity of A is the maximum 1-norm of any element in the Graver basis of the Graver basis of A.

$$||\mathbf{u}||_1 = |u_1| + |u_2| + \cdots + |u_m|.$$

Markov complexity

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Graver basis of A

The Graver basis of A has 15 elements

1 -1 0 -1 1 0 0 0 0
1 -1 0 0 0 0 -1 1 0
0 0 0 1 0 -1 -1 0 1
1 0 -1 -1 0 1 0 0 0
0 0 0 1 -1 0 -1 1 0
1 0 -1 0 0 0 -1 0 1
1 -1 0 -1 0 1 0 1 -1
0 0 0 0 1 -1 0 -1 1
0 1 -1 0 0 0 0 -1 1
1 0 -1 -1 1 0 0 -1 1
0 1 -1 0 -1 1 0 0 0
0 1 -1 1 -1 0 -1 0 1
0 1 -1 -1 0 1 1 -1 0
1 0 -1 0 -1 1 -1 1 0
1 -1 0 0 1 -1 -1 0 1

Graver complexity

The Graver basis of the Graver basis of A has 853 elements. The element

$$(0,0,0,0,3,0,-1,0,0,0,0,-2,2,0,1)$$

has the maximum 1-norm which is 9, so the Graver complexity of A is 9.

Graver complexity

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem

Kahle

The Graver complexity of B is 27.

Markov complexity

Theorem

Sturmfels and Santos (2003)
The Graver complexity of A is the maximum 1-norm of any element in the Graver basis of the Graver basis of A.

Theorem

Sturmfels and Santos (2003)
The Markov complexity of A is finite.

Markov complexity

Up to now, no formula for $m(A)$ (the Markov complexity of A) is known in general and there are only a few classes of toric ideals for which $m(A)$ has been computed.

Question: What is the Markov complexity of monomial curves in $\mathbb{A}^3$? (or what is the Markov complexity of $A = (n_1\ n_2\ n_3)$?)

Markov complexity for monomial curves in $\mathbb{A}^3$ which are not complete intersections.

Suppose that $A = \{n_1, n_2, n_3\} \subset \mathbb{Z}_{>0}$ such that I_A is not a complete intersection ideal. For $1 \leq i \leq 3$ we let c_i be the smallest element of $\mathbb{Z}_{>0}$ such that $c_i n_i = r_{ij} n_j + r_{ik} n_k$, $r_{ij}, r_{ik} \in \mathbb{Z}_{>0}$ with $\{i, j, k\} = \{1, 2, 3\}$.

Theorem

(Herzog 1970) Let $A = \{n_1, n_2, n_3\}$ be a set of positive integers with $\gcd(n_1, n_2, n_3) = 1$, and with the property that I_A is not a complete intersection ideal. Let $\mathbf{u}_1 = (-c_1, r_{12}, r_{13})$, $\mathbf{u}_2 = (r_{21}, -c_2, r_{23})$, $\mathbf{u}_3 = (r_{31}, r_{32}, -c_3)$. Then A has a unique minimal Markov basis, $\mathcal{M}(A) = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$.

Let $A = \{n_1, n_2, n_3\}$ be such that I_A is not a complete intersection. Let $r \geq 3$ and let T be the subset of containing all vectors of type 2 whose nonzero rows are of the form $\mathbf{u}, -\mathbf{u}$, with $\mathbf{u}$ in the Graver basis of A and all vectors of type 3 whose nonzero rows are permutations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Moreover $|T| = k\binom{r}{2} + 6\binom{r}{3}$, where k is the cardinality of the Graver basis of A.

Theorem

Let $A = \{n_1, n_2, n_3\}$ be such that I_A is not a complete intersection. Then the Markov complexity of A is 3. Moreover, for any $r \geq 3$ we have a unique minimal system of generators of cardinality $k\binom{r}{2} + 6\binom{r}{3}$, where k is the cardinality of the Graver basis of A.

Markov complexity for monomial curves in $\mathbb{A}^3$ which are complete intersections

Suppose that $A = \{n_1, n_2, n_3\} \subset \mathbb{Z}_{>0}$ is a monomial curve such that I_A is a complete intersection ideal. For $1 \leq i \leq 3$ we let c_i be the smallest element of $\mathbb{Z}_{>0}$ such that $c_i n_i = r_{ij} n_j + r_{ik} n_k$, $r_{ij}, r_{ik} \in \mathbb{N}$ with $\{i, j, k\} = \{1, 2, 3\}$. Herzog in 1970 shows that either $(0, -c_2, c_3) \in M(A)$ or $(c_1, 0, -c_3) \in M(A)$ or $(-c_1, c_2, 0) \in M(A)$. We recall the description of the universal Markov basis of A when $(0, -c_2, c_3) \in M(A)$.

Proposition

Let $A = \{n_1, n_2, n_3\}$ *be a set of positive integers such that* $\gcd(n_1, n_2, n_3) = 1$*,* I_A *is a complete intersection and* $(0, -c_2, c_3) \in M(A)$*. Let* $\mathbf{u}_1 = (-c_1, r_{12}, r_{13})$ *and* $\mathbf{u}_2 = (0, -c_2, c_3)$*. The universal Markov basis of* A *is*

$$\mathcal{M}(A) = \{\mathbf{u}_2, d \cdot \mathbf{u}_2 + \mathbf{u}_1 : \; -\lfloor \frac{r_{13}}{c_3} \rfloor \leq d \leq \lfloor \frac{r_{12}}{c_2} \rfloor\}.$$

Markov complexity for monomial curves in $\mathbb{A}^3$ which are complete intersections

Theorem

Let $A = \{n_1, n_2, n_3\}$ be such that I_A is a complete intersection. Then the Markov complexity of A is 2. Moreover, for any $r \geq 2$ we have a unique minimal system of generators of cardinality $k\binom{r}{2}$, where k is the cardinality of the Graver basis of A.

Graver complexity of monomial curves in $\mathbb{A}^3$

The next Theorem gives a lower bound for the Graver complexity of a monomial curve A in A^3.

Theorem

Let $A = \{n_1, n_2, n_3\}$ such that $\gcd(n_1, n_2, n_3) = 1$ and $d_{ij} = \gcd(n_i, n_j)$ for all $i \neq j$. Then

$$g(A) \geq \frac{n_1}{d_{12}d_{13}} + \frac{n_2}{d_{12}d_{23}} + \frac{n_3}{d_{13}d_{23}}.$$

In particular, if n_1, n_2, n_3 are pairwise prime then $g(A) \geq n_1 + n_2 + n_3$.

This shows that in general the upper bound for Markov complexity is rather crude: given any $k \in \mathbb{N}$, one can find appropriate $A = \{n_1, n_2, n_3\}$ so that the $g(A) \geq k$, while $m(A) \leq 3$.

Examples

(a) Let $A = \{3, 4, 5\}$. Computations with 4ti2 show that the maximum 1-norm of the elements of the Graver basis of the Graver basis of A is 12 and thus the Graver complexity of A equals the the lower bound $3 + 4 + 5$ of the Theorem.
(b) Let $A = \{2, 3, 17\}$. Computations with 4ti2 show that the maximum 1-norm of the elements of the Graver basis of the Graver basis of A is 30 and thus the Graver complexity of A is 30, while the lower bound of the Theorem is $22 = 2 + 3 + 17$.

Computing Markov complexity is an extremely challenging problem, and a formula for it seems hard to find in general.

Thank you