Hilbert function of numerical semigroup rings.

Grazia Tamone

DIMA - University of Genova - Italy

tamone@dima.unige.it

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Subject of the talk

We study the behaviour of the Hilbert function H_R of a one dimensional complete local ring R associated to a numerical semigroup $S \subseteq \mathbb{N}$, with a particular focus on the possible decrease of this function. After the basic definitions, we proceed by several steps:

- survey of rings R having the associated graded ring Cohen Macaulay: it is well-known that in these cases the function H_R does not decrease
- overview on some other classes of rings with H_R non decreasing
- focus on the question of finding conditions on S in order to have decreasing Hilbert function: recent results
- a description of classes of Gorenstein rings with H_R non decreasing.

Hilbert function for local rings

We recall the definition of the Hilbert function of a local ring.

Definition

Let $(R, \mathfrak{m}, k)$ be a noetherian d-dimensional local ring, the associated graded ring of R with respect to $\mathfrak{m}$ is

$$G := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$$

The Hilbert function $H_R : \mathbb{N} \longrightarrow \mathbb{N}$ of R is defined by means of the associated graded ring G:

$$H_R(n) := dim_k(\mathfrak{m}^n / \mathfrak{m}^{n+1})$$

While the Hilbert function of a Cohen Macaulay graded standard k-algebra is well understood, in the local case very little is still known. There are properties that cannot be carried on G: if R is Cohen Macaulay or even Gorenstein, in general G can be non Cohen Macaulay.

Semigroups rings

This talk deals with the Hilbert function of one dimensional semigroup rings. We recall the definition.

Let S be a *numerical semigroup* minimally generated by $\{n_1, n_2, \ldots, n_\nu\}$ where $n_1 < n_2 < \cdots < n_\nu$ and $GCD\{n_1, n_2, \ldots, n_\nu\} = 1$.

Classically S is associated to the rational affine monomial curve $C \subset \mathbb{A}_k^\nu$, parametrized by $x_i = t^{n_i}$, for $i = 1, ..., \nu$. The coordinate ring of C is $k[t^{n_1}, \ldots, t^{n_\nu}]$. C has only one singular point, the origin O, with local ring

$$\mathcal{O}_{C,O} = k[t^{n_1}, \ldots, t^{n_\nu}]_{(t^{n_1}, \ldots, t^{n_\nu})}$$

Definition

We call *semigroup ring associated to S* *the local ring*

$$R = k[[S]] := k[[t^{n_1}, \ldots, t^{n_\nu}]]$$

- R is the completion of $\mathcal{O}_{C,O}$
- R is isomorphic to $k[[X_1, \ldots, X_\nu]]/I$ where I, the defining ideal of C, is generated by binomials.

Semigroups: basic definitions

Given a numerical semigroup $S = \langle n_1, n_2, \cdots n_\nu \rangle$, let $R = k[[S]]$:

- denote the integer n_1 by e, the *multiplicity* of S and of R
  the integer ν is called the *embedding dimension* of S and of R
- $\mathfrak{m}$ and $M := S \setminus \{0\}$ are respectively the *maximal ideal* of R and of S

Let $v : k((t)) \longrightarrow \mathbb{Z} \cup \{\infty\}$ be the usual valuation given by the degree in t :

- $v(R) = S, \quad v(\mathfrak{m}) = M$
- for $n \in \mathbb{N}, \quad v(\mathfrak{m}^n) = nM = M + \cdots + M$ (n times)
- for any pair of nonzero fractional ideals $I \supseteq J$ of R it is possible to compute the length of the R-module I/J by means of valuations:

$$\ell_R(I/J) = |v(I) \setminus v(J)|$$

Apéry set and type

- The *Apéry set* (with respect to e) of S is
  $\text{Apéry}(S) := \{n \in S \mid n - e \notin S\}$ (shortly denoted by Apéry)
  the set of the smallest elements in S in each congruence class mod e.
- The *Frobenius number f* is the greatest element in $\mathbb{N} \setminus S$.
- The *Cohen Macaulay type* of R is $\tau(R) := \ell_R(R :_K \mathfrak{m}/R)$ where K is the fraction field of R.
- R is called *Gorenstein ring* if $\tau(R) = 1$,
  equivalently, the semigroup is *symmetric*: $n \in S \iff f - n \notin S$,
  equivalently, for each $n \in$ Apéry there exists $n' \in$ Apéry such that $n' + n = e + f$, the greatest element in Apéry.

Cohen Macaulay property of G

In the sequel we shall assume k an infinite field. First we discuss a relevant deeply studied question: the *Cohen Macaulayness* of G.

For a one dimensional local ring $(R, \mathfrak{m}, k)$ with k infinite there exists an element $x \in \mathfrak{m}$ such that $x\mathfrak{m}^n = \mathfrak{m}^{n+1}$, for $n >> 0$ (*superficial element*).
We denote by R' the quotient ring $R' = R/xR$.
For $a \in R$, let a^* be its image in G (the *initial form of a*).
We have the well-known theorem

Theorem

1. *The following conditions are equivalent*
   - *G is Cohen Macaulay*
   - *x^* is a non-zero divisor in G*
   - $H_R(n) - H_R(n-1) = H_{R'}(n)$ *for each* $n \geq 1$
2. *If G is Cohen Macaulay, then H_R is non-decreasing.*

Cohen Macaulay property of G

We recall sufficient conditions to have the Cohen Macaulayness of G: some results hold under more general assumptions (this list is not all-inclusive).

In the following cases the associated graded ring of R is Cohen Macaulay.

- $e \leq 3$ *or* $\nu = e$ *(maximal embedding dimension)* [Sally, 1977]
- *R Gorenstein with* $\nu = e - 2$ [Sally, 1980]
- $\nu = e - 1$ *and* $\tau(R) < e - 2$ [Sally, 1983]
- *The embedding dimension of S is four, under some other arithmetical conditions*
  [F.Arslan, P.Mete, M.Şahin, N.Şahin, several papers]

Cohen Macaulay property of G

- *In most cases when S is generated by an almost arithmetic sequence i.e., $\nu - 1$ generators are an arithmetic sequence,*
  [Molinelli, Patil -T, 1998]
- *S is obtained by particular techniques of gluing of semigroups*
  [Arslan, Mete, M.Şahin, 2009] [Jafari, Zarzuela, 2014]
- *S is generated by a generalized arithmetic sequence i.e.*
  $n_i = hn_1 + (i-1)d$, *with* $d, h \geq 1,\ 2 \leq i \leq \nu,\ GCD(n_1, d) = 1$
  (when $h = 1$, S is generated by an arithmetic sequence)
  [Sharifan, Zaare-Nahandi, 2009]

Example: $S = \langle 7, 17, 20, 23, 26 \rangle = \langle 7, 14{+}d, 14{+}2d, 14{+}3d, 14{+}4d \rangle$
$(h = 2,\ d = 3)$

The semigroup case

If $R = k[[S]]$ is a semigroup ring, the Cohen Macaulayness of G and the behaviour of the Hilbert function of R have also an handy characterisation by means of the semigroup S: we recall some tools.

Definition

For each $s \in S$, the *order of s is* $ord(s) := max\{h \in \mathbb{N} \mid s \in hM\}$

If $s \in S$ and $ord(s) = k$, then $(t^s)^* \in \mathfrak{m}^k/\mathfrak{m}^{k+1} \hookrightarrow G$
Note that if $s, s' \in S$ then:

$$(t^s)^*(t^{s'})^* \neq \overline{0} \ \ in \ G \iff ord(s) + ord(s') = ord(s+s')$$

Further, for a semigroup ring with multiplicity e, the element $x = t^e$ is *superficial*, hence by the above cited results:

Theorem

Let $R = k[[S]]$. The following conditions are equivalent:

1. *G is Cohen Macaulay*
2. $ord(s + ce) = ord(s) + c$ *for each* $s \in S$, $c \in \mathbb{N}$.

An easy example is the following.

Example

In $R = k[[t^7, t^9, t^{20}]]$ the initial form $(t^7)^*$ is a zero-divisor in G: in fact

$$ord(20 + 7) = ord(27) = 3 > ord(20) + 1$$

and so G is not Cohen Macaulay.

For semigroup rings the Apèry set is an useful tool:

Proposition

Let $R = k[[S]]$, $R' = R/t^e R$ *and let* $Ap_n := \{s \in Apéry(S) \mid ord(s) = n\}$.

- $H_R(n) = |nM \setminus (n+1)M| = |\{s \in S \mid ord(s) = n\}|$
- $H_{R'}(n) = |Ap_n|$
- G *is Cohen Macaulay* $\iff H_R(n) - H_R(n-1) = |Ap_n|, \quad \forall\, n \geq 1$
  (*recall:* G *is Cohen Macaulay* $\iff H_R(n) - H_R(n-1) = H_{R'}(n)$).

Behaviour of the Hilbert function

In general, when G is not Cohen Macaulay, the function H_R can be decreasing or not:

Definition

The Hilbert function of R is said to be decreasing if there exists $n \in \mathbb{N}$ such that

$$H_R(n) < H_R(n-1)$$

in this case we say that H_R decreases at level n.

Examples

Example

Let $R = k[[S]]$ with $S = \langle 6, 7, 15, 23 \rangle$.
First note that $ord(15 + e) = ord(15 + 6) = ord(21) = 3 > ord(15) + 1$, then G is not Cohen Macaulay.

One can compute that $H_R = [1, 4, 4, 5, 5, 6 \rightarrow]$ is non-decreasing.

$\text{Apéry}(S) = \{0, 7, 14, 15, 22, 23\}$, $Ap_1 = \{7, 15, 23\}$, $Ap_2 = \{14, 22\}$
hence $H_{R'} = [1, 3, 2]$.

Example

Let $R = k[[S]]$, with $S = \langle 13, 19, 24, 44, 49, 54, 55, 59, 60, 66 \rangle$

First note that $ord(44 + e) = ord(57) = 3 > ord(44) + 1$, then G is not Cohen Macaulay. One can verify that H_R decreases at level 2:

$$H_R = [1, 10, 9, 11, 12, 13 \rightarrow]$$

Further $Ap_2 = \{38, 43, 48\}$, $H_{R'} = [1, 9, 3]$.

Non-decreasing Hilbert function

Under several assumptions we know that $R = k[[S]]$ has non decreasing Hilbert function. In particular this fact is true if

- *G is Cohen Macaulay*
- $\nu \leq 3$ *or* $\nu \leq e \leq \nu + 2$ [Sally, Elías, Rossi - Valla]
- *S is generated by an almost arithmetic sequence* [T, 1998]
- *S is balanced, i.e.* $n_i + n_j = n_{i-1} + n_{j+1}$, *for each* $i \neq j \in [2, \nu - 1]$ [Patil -T, 2011], [Cortadellas, Jafari, Zarzuela, 2013]
- *S is obtained by particular techniques of gluing of semigroups* [Arslan, Mete, M.Şahin, 2009] [Jafari, Zarzuela, 2014]
- *R is Gorenstein* with $\nu = 4$ and

  S satisfies some arithmetic conditions [Arslan, Mete, 2007]
  or S is constructed by gluings [Arslan, Sipahi, N.Şahin, 2013].

Decreasing H-function: main tools

Now we want to describe conditions on the semigroup S in order to obtain rings with decreasing Hilbert function: we need some definitions and facts.

Definition

- a *maximal representation* of $s \in S$ is any expression

  $s = \sum_{j=1}^{\nu} a_j n_j, \ a_j \in \mathbb{N}, \text{ with } \ \sum a_j = ord(s)$

- the *support* of (a maximal representation of) $s \in S$ is

  $Supp(s) := \{n_j \mid a_j \neq 0\}$

- For a subset $X \subset \mathbb{N}$ define $Supp(X) := \cup_{x \in X} Supp(x)$.

Decrease of the H-function

Since $H_R(n) = |\{s \in S \mid ord(s) = n\}|$ we consider the following subsets :

$$S_n := \{s \in S \mid ord(s) = n\} =$$

$$= \{s' + e \in S_n \mid s' \in S_{n-1}\} \cup \{t + e \in S_n \mid ord(t) \le n-2\} \cup Ap_n$$

$$S_{n-1} = \{s' \in S_{n-1} \mid s' + e \in S_n\} \cup \{s' \in S_{n-1} \mid ord(s' + e) > n\}$$

$$C_n := \{s \in S_n \mid s - e \notin S_{n-1}\} = \{t + e \in S_n \mid ord(t) \le n-2\} \cup Ap_n$$

$$D_n := \{ s' \in S_{n-1} \mid ord(s' + e) > n\}, \text{ for } n \ge 2, \quad D_1 = \emptyset$$

D_n = set of elements of S that "skip" the order when adding e.

Proposition

- $H_R(n) - H_R(n-1) = |S_n| - |S_{n-1}| = |C_n| - |D_n|$ *for each* $n \ge 1$.
- G *is Cohen Macaulay* $\iff D_n = \emptyset$ *for each* n.
- H_R *decreases at level* $n \iff |C_n| < |D_n|$.

Proposition

1. $C_1 = Ap_1, \quad C_2 = Ap_2.$
2. [Patil -T, 2011] *For* $s = \sum_{i=1,\ldots,\nu} a_i n_i \in C_k$ *(maximal representation with* $\sum a_i = k$*), and for each choice* $0 \le b_i \le a_i$, $i \in [1,\nu]$ *with* $\sum b_i = h$,

   the "induced" element $s' = \sum_{i=1,\ldots,\nu} b_i n_i$ *belongs to* C_h.

Corollary

Let $k \ge 2$:

1. $Supp(C_k) \subseteq Supp(Ap_2)$
2. $Supp(D_k + e) \subseteq Supp(Ap_2)$
3. *In particular* $Supp(Ap_k) \subseteq Supp(Ap_2)$

Proposition

[D'Anna, Di Marca, Micale, 2015]:

1. *If* $|D_k| \leq k+1$ *for every* $k \geq 2$, *then* H_R *is non-decreasing*
2. *If* $|D_k| > k+1$, *then* $|C_h| \geq h+1$ *for all* $h \in [2,k]$
3. *If* H_R *decreases, then* $|C_2| = |Ap_2| \geq 3$.

For $k = 2$ the above proposition doesn't give informations on $|C_3|$: a bound is specified in part 1 of the next result. This information will be very useful in the sequel. The proof requires many technical computations.

Proposition

If H_R *is decreasing then*

1. $|C_3| \geq 4$
2. *If* $|Ap_2| = 3$ *there exist* $n_i, n_j \in Ap_1$ *such that*

$$Ap_2 = \{2n_i, n_i + n_j, 2n_j\}$$

Example

By the above cited results, H_R decreasing implies $e \geq \nu + 3$. The "smallest" known example with $e = \nu + 3$ ($e = 13$, $\nu = 10$) is:

Example

$R = k[[S]]$, where $S = \langle 13, 19, 24, 44, 49, 54, 55, 59, 60, 66 \rangle$

$$H_R = [1, 10, 9, 11, 12, 13 \rightarrow]$$

$$\text{Apéry}(S) = \{\ 0, 19, 24, 38, 43, 44, 48, 49, 54, 55, 59, 60, 66\ \}$$

$M \setminus 2M =$	13	19	24			44		49	54	55	59	60	66
$2M \setminus 3M =$	26	32	37	38	43		48			68		73	79

$$D_2 = \{44, 49, 54, 59\} \qquad C_2 = Ap_2 = \{38, 43, 48\} = \{19 \cdot 2, 19 + 24, 24 \cdot 2\}$$

$$D_2 + e = \{57, 62, 67, 72\} \qquad 57 = 3 \cdot 19, \quad 62 = 2 \cdot 19 + 24, \quad 67 = 19 + 2 \cdot 24, \quad 72 = 3 \cdot 24$$

$$D_3 = \{68, 73\} \qquad C_3 = \{57, 62, 67, 72\} = D_2 + e$$

[Molinelli -T, 1999]

Case $e = \nu + 3$

If $e = \nu + 3$, by Macaulay's theorem, the possible Hilbert functions of $R' = R/t^e R$ are $[1, \nu - 1, 3]$ $[1, \nu - 1, 2, 1]$ $[1, \nu - 1, 1, 1, 1]$

As seen above, H_R decreasing implies $|Ap_2| \geq 3$ and so $H_{R'} = [1, \nu - 1, 3]$.

Theorem

[O -T, 2016] *Let $e = \nu + 3$. The following conditions are equivalent:*

1. H_R *decreases*
2. H_R *decreases at level 2*
3. $H_{R'} = [1, \nu - 1, 3]$ *and there exist* $n_i \neq n_j \in Ap_1$ *such that*
   - $Ap_2 \quad = \{2n_i, n_i + n_j, 2n_j\}$
   - $D_2 + e = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$

Further if the above conditions hold, then $e \geq 13$.

Corollary

When $e = 13 = \nu + 3$*:*

$$H_R \text{ decreases} \iff Ap(S) = \left[\begin{array}{l} n_i, n_j \\ 2n_i, n_i + n_j, 2n_i \\ 3n_i - e, 2n_i + n_j - e, n_i + 2n_j - e, 3n_j - e \\ 3n_i + n_j - \alpha e,\ 2n_i + 2n_j - \beta e, \\ 3n_i + 2n_j - \gamma e \end{array}\right.$$

for suitable α, β, γ *and* $\left[\begin{array}{ccc} \text{either} & n_j & = 4n_i (mod\, 13) \\ \text{or} & n_j & = 10n_i\, (mod\, 13)\,. \end{array}\right.$

Example

For $S = \langle 13, 19, 24, 44, 49, 54, 55, 59, 60, 66 \rangle$ (considered before)

$$n_i = 19, \quad n_j = 24 \equiv 76 = 4n_i\, (mod\, 13),$$

$$\alpha = 2, \quad \beta = 2, \quad \gamma = 3.$$

Case $e = \nu + 4$

As in case $e = \nu + 3$, we deduce that H_R decreasing implies that the Hilbert function of $R' = R/t^e R$ can be

either $[1, \nu - 1, 3, 1]$ or $[1, \nu - 1, 4]$

Theorem

[O -T, 2016] *Let* $e = \nu + 4$, $|Ap_2| = 3$, $|Ap_3| = 1$.
The following conditions are equivalent:

1. H_R *decreases*
2. H_R *decreases at level* $\ell \leq 3$
3. *there exist* $n_i \neq n_j \in Ap_1$ *such that*
   - $Ap_2 = \{2n_i, n_i + n_j, 2n_j\}$
   - $C_3 = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$
   - $D_\ell + e = \{4n_i, 2n_i + n_j, n_i + 2n_j, 3n_j\}$ *if* $\ell = 2$
     $D_\ell + e = \{(\ell + 1)n_i, \ell n_i + n_j, \dots, (\ell + 1)n_j\}$ *if* $\ell = 3$

Example

We show two examples for $e = \nu + 4$ with $\ell = 2$ and $\ell = 3$.

Example

1. Let $S =< 17, 19, 22, 43, 45, 46, 47, 48, 49, 50, 52, 54, 59 >$
   $n_i = 19, n_j = 22, \ \ \nu = 13 = e - 4, \ Ap_2 = \{38, 41, 44\},$
   $Ap_3 = \{57 \ = 3n_i\},$
   $D_2 + e = \{76 = 4n_i, \ 60 = 2n_i + n_j, \ 63 = n_i + 2n_j, \ 66 = 3n_j\};$
   $\ell = 2, \quad H_R = [1, 13, 12, 13, 15, 16, 17 \rightarrow].$
2. Let $S = \langle 19, 21, 24, 46, 47, 49, 50, 51, 52, 53, 54, 55, 56, 58, 60\rangle$
   $n_i = 21, n_j = 24, \ \ e = \nu + 4,$
   $Ap_2 = \{42, 45, 48\}, Ap_3 = \{63 \ = 3n_i\},$
   $C_3 = \{66, 69, 72\} \cup \{63\},$
   $D_3 + e = \{4n_i, 3n_i + n_j, 2n_i + 2n_j, n_i + 3n_j, 4n_j\};$
   $\ell = 3, \ \ H_R = [1, 15, 15, 14, 16, 18, 19 \rightarrow].$

When $e = \nu + 4$, the remaining case with H_R decreasing has $H_{R'} = [1, \nu - 1, 4]$: we have an explicit description of the Apéry set of S and

Theorem

[O -T, 2016] *Assume $e = \nu + 4$, $|Ap_2| = 4$, $Ap_3 = \emptyset$. Are equivalent:*

1. *H_R decreases at level 2.*
2. *There exist $n_i, n_j, n_k \in Ap_1$, distinct elements, such that*

$$\text{either} \quad \begin{cases} Ap_2 & = \{2n_i, n_i + n_j, 2n_j, n_i + n_k\} \\ C_3 & = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 2n_i + n_k\} \end{cases}$$

$$\text{or} \quad \begin{cases} Ap_2 & = \{2n_i, n_i + n_k, 2n_j, 2n_k)\} \\ C_3 & = \{3n_i, 2n_i + n_j, n_i + 2n_j, 3n_j, 3n_k\} \end{cases}$$

Example

Let $S = \langle 17, 19, 22, 31, 40, 42, 43, 45, 46, 47, 49, 52, 54 \rangle$, $\nu = e - 4$, $n_i = 19, n_j = 22, n_k = 31$, $Ap_2 = \{38, 41, 44, 50\} = \{2n_i, n_i + n_j, 2n_j, n_i + n_k\}$, $Ap_3 = \emptyset$, $H_R = [1, 13, 12, 14, 16, 17 \rightarrow]$.

Hilbert function for certain Gorenstein rings

Theorem

[O -T, 2016] *If $R = k[[S]]$ is a Gorenstein semigroup ring with $e \leq \nu + 4$, then the Hilbert function H_R is non decreasing.*

Proof.

First recall that by the above cited Sally's results,for any local one-dimensional Gorenstein ring with $e \leq \nu + 2$ the associated graded ring G is Cohen Macaulay and so H_R is non decreasing.
If $\nu + 3 \leq e \leq \nu + 4$, by the above arguments the only possible shape of the Hilbert function $H_{R'}$ compatible with the decrease of H_R and the symmetry of S is $[1, \nu - 1, 3, 1]$, (with $e = \nu + 4$). In this case, the particular structure of Apéry(S) and of D_2 allow to prove that S cannot be symmetric. This theorem is a contribution to the following problem

Is the Hilbert function of a Gorenstein one-dimensional local ring non-decreasing?

Thanks for your attention!

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