Symmetric numerical semigroups with decreasing Hilbert function

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Joint work with A. Oneto and G. Tamone

We use the following notations:

- S is a numerical semigroup;
- $M(S) = S \setminus \{0\}$ is the *maximal ideal* of S;
- $e(S) = \min(M(S))$ is the *multiplicity* of S;
- $f(S) = \max\{\mathbb{N} \setminus S\}$ is the *Frobenius number* of S;
- $K(S) = \{x \in \mathbb{N} \mid f(S) - x \notin S\}$ is the *standard canonical ideal* of S. We call *canonical ideals* all the relative ideals $K(S) + x$ for any $x \in \mathbb{Z}$;
- $\mathrm{PF}(S) = \{x \in \mathbb{Z} \setminus S \mid x + s \in S \text{ for any } s \in M(S)\}$ is the set of the *pseudo-Frobenius numbers* of S;
- $t(S) = |\mathrm{PF}(S)|$ is the *type* of S;
- $H_S(i) = |iM(S) \setminus (i+1)M(S)|$ is the i-th value of the Hilbert function of S. Here $iM(S)$ is the sum $M(S) + M(S) + \cdots + M(S) \quad (i \text{ times})$;
- We write the *Hilbert function* of S as $[H_S(0), H_S(1), \ldots, H_S(n) \rightarrow]$, where the arrow means that all the values greater than n are equal to $H_S(n)$.
- $\nu(S) = H_S(1)$ is the *embedding dimension* of S;

Rossi's problem

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Problem (M.E. Rossi)

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We recall that a numerical semigroup is said to be symmetric if $S = K(S)$ or equivalently if it has type 1. If we restrict to numerical semigroup rings, we can rewrite the previous problem as follows.

Problem

Is the Hilbert function of a symmetric numerical semigroup not decreasing?

In the last ten years several authors gave a positive answer in some particular cases:

- F. Arslan, P. Mete and M. Şahin [2009], for infinitely many families obtained using the notion of nice gluing of numerical semigroups;
- R. Jafari and S. Zarzuela Armengou [2014], for some families of numerical semigroups through the concept of gluing;
- A. Oneto and G. Tamone [2016], when $\nu(S) \geq e(S) - 4$.

Moreover the answer is positive also for several families of symmetric numerical semigroups with embedding dimension 4:

- F. Arslan and P. Mete [2007];
- D.P. Patil and G. Tamone [2011];
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However the answer is negative in general.

Given a proper ideal E of S and an odd integer $b \in S$, the numerical duplication of S with respect to E and b is defined as the numerical semigroup

$$S \bowtie^b E = 2 \cdot S \cup (2 \cdot E + b),$$

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where $2 \cdot X = \{2x \mid x \in X\} \neq 2X$ for any set X. Equivalently, if $S = \langle s_1, \ldots, s_\nu \rangle$ and $E = \langle n_1, \ldots, n_h \rangle$, we have

$$S \bowtie^b E = \langle 2s_1, \ldots, 2s_\nu, 2n_1 + b, \ldots, 2n_h + b \rangle$$

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Proposition (D'Anna, S.)

The numerical semigroup $S \bowtie^b E$ is symmetric if and only if E is a canonical ideal of S.

The Hilbert function of the numerical duplication

Let T be the numerical semigroup $S \ltimes^b E$.

Proposition (Barucci, D'Anna, S.)

For any $i > 0$ the i-value of the Hilbert function of T is

$$H_T(i) = H_S(i) + |((i-1)M(S) + E) \setminus ((i-2)M(S) + E)|.$$

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The numerical semigroup S is said to be *almost symmetric* if $M(S) + K(S) = M(S)$. In this case the formula above becomes easier:

$$H_T(0) = 1,$$
$$H_T(1) = \nu(S) + t(S),$$
$$H_T(i) = H_S(i) + H_S(i-1) \text{ if } i \geq 2.$$

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Corollary

Let S be almost symmetric and let E be a canonical ideal of S. If $H_S(i-1) > H_S(i+1)$, then $H_T(i) > H_T(i+1)$. In particular T is a symmetric numerical semigroup with decreasing Hilbert function.

Question

Are there almost symmetric numerical semigroups S such that $H_S(i-1) > H_S(i+1)$ for some i?

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$$S = \langle 30, 35, 42, 47, 108, 110, 113, 118, 122, 127, 134, 139 \rangle$$

is almost symmetric and its Hilbert function is $H_S = [1, 12, \mathbf{17}, \mathbf{16}, 25, 30 \rightarrow]$. Therefore H_S decreases, but $H_S(i-1) \leq H_S(i+1)$ for any i.

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The same happens for the almost symmetric semigroup

$$S = \langle 56, 63, 72, 79, 271, 273, 275, 278, 282, 285, 289, 291, 298, \\ 304, 305, 307, 311, 314, 318, 320, 321, 322, 325, 332 \rangle$$

that has Hilbert function $[1, \mathbf{24}, \mathbf{23}, \mathbf{27}, \mathbf{25}, 36, 49, 56 \rightarrow]$.

Definition

1. If s is an element of S, the *order* of s is $\mathsf{ord}(s) := \max\{i \mid s \in iM(S)\}$.
2. The Apéry set of S is $\mathrm{Ap}(S) := \{s \in S \mid s - e(S) \notin S\}$.
3. $\mathrm{Ap}_k(S) := \{s \in \mathrm{Ap}(S) \mid \mathsf{ord}(s) = k\}$.
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If S has decreasing Hilbert function, D'Anna, Di Marca, and Micale proved that $|\mathrm{Ap}_2(S)| \geq 3$. So we first consider the simpler case:

Proposition (Oneto, S., Tamone)

Assume that $|\mathrm{Ap}_2(S)| = 3$, $\mathrm{Ap}_k(S) = \emptyset$ for all $k \geq 3$ and H_S is decreasing. Then S is not almost symmetric.

If H_S is decreasing we denote by ℓ the minimum level in which decreases. Moreover we set $d = \max\{\mathrm{ord}(s) \mid s \in \mathrm{Ap}(S)\}$.

Proposition (Oneto, S., Tamone)

Assume that $|\mathrm{Ap}_2(S)| = 3$, $|\mathrm{Ap}_3(S)| = 1$ and H_S is decreasing.

1. *If S is almost symmetric, then $\ell \geq 3$.*

2. *If $\ell \geq 3$, then $H_S(h) = H_S(\ell - 1)$ for all $h \in [1, \ell - 1]$. Further $H_S(\ell - 2) - H_S(\ell) = 1$.*

Therefore it is enough to find an almost symmetric numerical semigroup with decreasing Hilbert function such that $|\mathrm{Ap}_2(S)| = 3$ and $|\mathrm{Ap}_3(S)| = 1$.

Come back to the previous talk

Proposition (Oneto, Tamone)

Assume that $|\mathrm{Ap}_2(S)| = 3$, $|\mathrm{Ap}_3(S)| = 1$, H_S *is decreasing and* $(\ell, d) \neq (3,3)$. *Then* $\ell \leq d$ *and there exist* $n_1, n_2 \in \mathrm{Ap}_1(S)$ *such that* $\mathrm{Ap}_k(S) = kn_1$, *for* $3 \leq k \leq d$ *and* $\mathrm{Ap}_2(S) = \{2n_1, n_1 + n_2, 2n_2\}$. *Moreover for* $2 \leq k \leq \ell - 1$

$$D_k + e(S) = \{kn_1 + n_2, (k-1)n_1 + 2n_2, \ldots, (k+1)n_2\},$$
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we get the following generators:

$$s_{p,q} := pn_1 + qn_2 - (p+q-2)e(S),$$
$$t_1 := (\ell+1)n_1 - (\ell-1)e(S),$$

where $0 \leq p \leq \ell$, $1 \leq q \leq \ell+1$ and $2 \leq p+q \leq \ell+1$.

To force that S is almost symmetric we use the following theorem:

Theorem (Nari)

Set $\mathrm{Ap}(S) = \{0 < \alpha_1 < \cdots < \alpha_m\} \cup \{\beta_1 < \cdots < \beta_{t(S)-1}\}$, *where* $\mathrm{PF}(S) = \{\beta_i - e(S) \,|\, 1 \leq i \leq t(S)-1\} \cup \{\alpha_m - e(S) = f(S)\}$ *and* $m = e(S) - t(S)$. *Then* S *is almost symmetric if and only if:*

1. $\alpha_i + \alpha_{m-i} = \alpha_m$ *for all* $i \in \{1, 2, \ldots, m-1\}$;

2. $\beta_j + \beta_{t(S)-j} = \alpha_m + e(S)$ *for all* $j \in \{1, 2, \ldots, t(S)-1\}$.

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In our case $\alpha_m = \ell n_1$. Clearly the elements $\{pn_1 \mid 1 \leq p \leq \ell\}$ satisfy the first condition. Further also $\{n_2\} \cup \{qn_2 - (q-2)e(S) \mid 2 \leq q \leq \ell + 1\}$ satisfy the conditions of the theorem. On the other hand, if $2 \leq p + q \leq \ell + 1$, $p \geq 1$ and $q \geq 1$, we require that among our generators there are also

$$r_{p,q} := \ell n_1 + e(S) - s_{p,q},$$
$$t_2 := \ell n_1 + e(S) - t_1 = \ell e(S) - n_1.$$

Since $\mathrm{Ap}(S) = \mathrm{Ap}_1(S) \cup \{2n_1, n_1 + n_2, 2n_2\} \cup \{kn_1 \mid 3 \leq k \leq \ell\}$, it follows that $\nu(S) = e(S) - \ell - 1$. Moreover in $\mathrm{Ap}_1(S)$ there are all the elements $\{e(S), n_1, n_2, t_1, t_2\}$, $\{s_{p,q}\}$ and $\{r_{p,q}\} \setminus \{n_1 + n_2, 2n_2\}$. Then, if we require that these elements are distinct, we get

$$e(S) \geq 5 + \frac{\ell^2 + 3\ell}{2} + \frac{\ell^2 + \ell}{2} - 2 + \ell + 1 = \ell^2 + 3\ell + 4.$$

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Construction (Oneto, S., Tamone)

Let $\ell \geq 4$ be an integer such that $\ell \notin \{14 + 22k, 35 + 46k \mid k \in \mathbb{N}\}$ and let $e := \ell^2 + 3\ell + 4$. Further we set

$$\begin{cases} n_1 := e + (2\ell - 1), & n_2 := e + (\ell^2 - 6), \quad \text{if } \ell \text{ is odd,} \\ n_1 := e + (\ell - 3), & n_2 := e + (\ell^2 - \ell - 6), \quad \text{if } \ell \text{ is even.} \end{cases}$$

We denote by S_ℓ the semigroup generated by $\{e, n_1, n_2, t_1, t_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\}$.

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- The Hilbert function of S_ℓ decreases at level ℓ and consequently

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$$H_{S_\ell} = [1, \nu, \nu, \dots, \nu, \nu - 1, H_{S_\ell}(\ell + 1), \dots];$$

- The type of S_ℓ is $t(S_\ell) = \nu - 1 = \ell^2 + 2\ell + 2$.

The "smallest" numerical semigroup that we have constructed is

$$S_4 = \langle 32, 33, 38, 69, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95 \rangle.$$

We have $\mathrm{Ap}_2(S_4) = \{66, 71, 76\}$, $\mathrm{Ap}_3(S_4) = \{99\}$, $\mathrm{Ap}_4(S_4) = \{132\}$ and its Hilbert function is $[1, 27, 27, \mathbf{27}, \mathbf{26}, 27, 29, 30, 31, 32 \rightarrow]$.

The "smallest" numerical semigroup that we have constructed is

$$S_4 = \langle 32, 33, 38, 69, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95\rangle.$$

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Moreover if we set $E := K(S) + 101 = K(S) + f(S) + 1 \subseteq S$, we get

$$S \bowtie^{33} E = \langle 64, 66, 76, 138, 144, 146, 148, 150, 154, 156, 158, 160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188, 190, 235, 309, 313, 315, 317, 319, 321, 323, 325, 327, 329, 331, 333, 335, 337, 339, 341, 343, 345, 347, 349, 351, 353, 355, 357, 361\rangle.$$

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This is a symmetric numerical semigroup and its Hilbert function is

$$[1, 53, 54, \mathbf{54}, \mathbf{53}, 53, 56, 59, 61, 63, 64 \rightarrow].$$

Theorem (Oneto, S., Tamone)

Let $m, \ell > 1$ *be integers such that* $\ell \notin \{14 + 22k, 35 + 46k \mid k \in \mathbb{N}\}$*. Then there exist infinitely many symmetric numerical semigroups* T *such that* $H_T(\ell - 1) - H_T(\ell) > m$*.*

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Consider $T_0 = S_5$ that has Hilbert function $[1, 38, 38, 38, \mathbf{38}, \mathbf{37}, 44 \rightarrow]$. All the following semigroups are almost symmetric:

- If $T_1 = T_0 \bowtie^{53} M(T_0)$, then $H_{T_1} = [1, 76, 76, 76, \mathbf{76}, \mathbf{74}, 88 \rightarrow]$;
- If $T_2 = T_1 \bowtie^{141} M(T_1)$, then $H_{T_2} = [1, 152, 152, 152, \mathbf{152}, \mathbf{148}, 176 \rightarrow]$;
- If $T_3 = T_2 \bowtie^{317} M(T_2)$, then $H_{T_3} = [1, 304, 304, 304, \mathbf{304}, \mathbf{296}, 352 \rightarrow]$;
- If $T_4 = T_3 \bowtie^{669} M(T_3)$, then $H_{T_4} = [1, 608, 608, 608, \mathbf{608}, \mathbf{592}, 704 \rightarrow]$;

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Let $m, \ell > 1$ be integers such that $\ell \notin \{14 + 22k, 35 + 46k \mid k \in \mathbb{N}\}$. Then there exist infinitely many symmetric numerical semigroups T such that $H_T(\ell - 1) - H_T(\ell) > m$.

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- If $T_3 = T_2 \bowtie^{317} M(T_2)$, then $H_{T_3} = [1, 304, 304, 304, \mathbf{304}, \mathbf{296}, 352 \rightarrow]$;
- If $T_4 = T_3 \bowtie^{669} M(T_3)$, then $H_{T_4} = [1, 608, 608, 608, \mathbf{608}, \mathbf{592}, 704 \rightarrow]$;
- Moreover if we set $K := K(T_4) + f(T_4) + 1 \subseteq T_4$, the numerical semigroup $T = T_4 \bowtie^{1373} K$ is symmetric and has Hilbert function

$$H_T = [1, 1215, 1216, 1216, \mathbf{1216}, \mathbf{1200}, 1296, 1408 \rightarrow].$$

Note that T has 1215 minimal generators included between 1408 and 23835.

Consider the almost symmetric numerical semigroup

$$T_0 = \langle 30, 33, 37, 64, 68, 69, 71, 72, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92 \rangle$$

that has Hilbert function $[1, 26, \mathbf{26}, \mathbf{25}, \mathbf{24}, 27, 28, 29, 30 \rightarrow]$ and type 25.

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that has Hilbert function $[1, 26, \mathbf{26}, \mathbf{25}, \mathbf{24}, 27, 28, 29, 30 \rightarrow]$ and type 25. Let K_i be a proper canonical ideal of T_i and b_i an arbitrary odd element of T_i. All the following numerical semigroups are symmetric.

- If $T_1 = T_0 \bowtie^{b_0} K_0$, then $H_{T_1} = [1, 51, \mathbf{52}, \mathbf{51}, \mathbf{49}, 51, 55, 57, 59, 60 \rightarrow]$;
- If $T_2 = T_1 \bowtie^{b_1} K_1$, its Hilbert function is

$$H_{T_2} = [1, 52, 103, \mathbf{103}, \mathbf{100}, 100, 106, 112, 116, 119, 120 \rightarrow];$$

- If $T_3 = T_2 \bowtie^{b_2} K_2$ has Hilbert function

$$H_{T_3} = [1, 53, 155, \mathbf{206}, \mathbf{203}, \mathbf{200}, 206, 218, 228, 235, 239, 240 \rightarrow];$$

- If $T_4 = T_3 \bowtie^{b_3} K_3$ has Hilbert function

$$H_{T_4} = [1, 54, 208, 361, \mathbf{409}, \mathbf{403}, 406, 424, 446, 463, 474, 479, 480 \rightarrow].$$

Consider the numerical semigroup

$$S := \langle 30, 33, 37, 64, 68, 69, 71, 72, 73, 75, \ldots, 89, 91, 92, 95 \rangle$$

that is not almost symmetric and has Hilbert function

$$[1, \mathbf{27}, \mathbf{26}, \mathbf{25}, \mathbf{24}, 27, 28, 29, 30 \rightarrow].$$

If we set $K := K(S) + 66 \subseteq S$, the semigroup $S \bowtie^{33} K$ is symmetric and has Hilbert function

$$[1, 54, 55, \mathbf{55}, \mathbf{54}, 57, 58, 59, 60 \rightarrow].$$

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If we set $K := K(S) + 66 \subseteq S$, the semigroup $S \bowtie^{33} K$ is symmetric and has Hilbert function

$$[1, 54, 55, \mathbf{55}, \mathbf{54}, 57, 58, 59, 60 \rightarrow].$$

In the previous talk it is showed that if T is a symmetric semigroup with decreasing Hilbert function, then $e(T) - \nu(T) \geq 5$; in this example $e(S \bowtie^{33} K) - \nu(S \bowtie^{33} K) = 6$.

Examples 3

If E is only a relative ideal it is still possible to define the numerical duplication. This is a numerical semigroup if and only if $E + E + b \subseteq S$. Moreover this is symmetric if and only if E is a canonical ideal.

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Example

Consider the numerical semigroup

$$S = \langle 30, 33, 37, 73, 76, 77, 79, 80, 81, 82, 83, 84, \\ 85, 86, 87, 88, 89, 91, 92, 94, 95, 98, 101, 108\rangle$$

that has Hilbert function $[1, 24, \mathbf{25}, \mathbf{24}, \mathbf{23}, 25, 27, 29, 30 \rightarrow]$. Then

- $T_1 := S \bowtie^{67} K(S)$ has Hilbert function $[1, 43, \mathbf{47}, \mathbf{45}, 49, 51, 60 \rightarrow]$;
- $T_2 := S \bowtie^{73} K(S)$ has Hilbert function $[1, \mathbf{44}, \mathbf{43}, \mathbf{41}, 49, 58, 60, \rightarrow]$;
- $T_3 := S \bowtie^{79} K(S)$ has Hilbert function $[1, \mathbf{44}, \mathbf{41}, \mathbf{40}, 52, 58, 60, \rightarrow]$;
- $T_4 := S \bowtie^{81} K(S)$ has Hilbert function $[1, 43, 45, 47, 52, 54, 56, 58, 60 \rightarrow]$;
- $T_5 := S \bowtie^{85} K(S)$ has Hilbert function $[1, \mathbf{44}, \mathbf{42}, 45, 52, 54, 58, 60 \rightarrow]$;
- $T_6 := S \bowtie^{87} K(S)$ has Hilbert function $[1, 46, \mathbf{48}, \mathbf{47}, 49, 51, 56, 58, 60 \rightarrow]$;
- $T_7 := S \bowtie^{93} K(S)$ has Hilbert function $[1, 47, \mathbf{49}, \mathbf{48}, 48, 50, 55, 58, 60 \rightarrow]$.

Example

Consider S_{15} that has 258 generators. According to GAP, the Hilbert function of the symmetric semigroup $S_{15} \bowtie^{957} K(S_{15})$ decreases 13 times

$[1, 514, \mathbf{514}, \mathbf{513}, \mathbf{512}, \mathbf{511}, \mathbf{510}, \mathbf{509}, \mathbf{508}, \mathbf{507}, \mathbf{506}, \mathbf{505}, \mathbf{504}, \mathbf{503}, \mathbf{502}, \mathbf{500}, \dots]$

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Among the symmetric semigroups with decreasing Hilbert function, the following is the semigroup with the smallest multiplicity and embedding dimension that I know.

Example

Consider $S = \langle 19, 21, 24, 47, 49, 50, 51, 52, 53, 54, 55, 56, 58, 60 \rangle$ and

$$S \bowtie^{49} K(S) = \langle 38, 42, 48, 49, 94, 100, 101, 102, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 115, 116, 117, 119, 120, 121, 123, 127 \rangle.$$

Even if S has non-decreasing Hilbert function $[1, 14, 14, 14, 16, 18, 19 \rightarrow]$, the Hilbert function of $S \bowtie^{49} K(S)$ is $[1, \mathbf{26}, \mathbf{25}, 25, 32, 38 \rightarrow]$.

Let R be a commutative ring with identity, let I be a proper ideal of R and let $a, b \in R$. Consider the Rees algebra $R[It] = \oplus_{n \geq 0} I^n t^n \subseteq R[t]$, where t is an indeterminate. Then we define the ring

$$R(I)_{a,b} = \frac{R[It]}{(t^2 + at + b)R[t] \cap R[It]}.$$

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Let $S = \langle s_1, \dots, s_\nu \rangle$ and let $R = k[[S]] = k[[x^{s_1}, \dots, x^{s_\nu}]]$ be its numerical semigroup ring. If m is odd and $b = x^m$, then $R(I)_{0,-b} \cong k[[S \bowtie^m v(I)]]$, where v is the standard valuation.

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The Hilbert function and the Gorenstein property of $R(I)_{a,b}$ are independent of a and b. This means that if we find a symmetric numerical semigroup $S \bowtie^b E$ with decreasing Hilbert function and $R(I)_{0,b}$ is isomorphic to its numerical semigroup ring, where $R = k[[S]]$, then $R(I)_{a,b}$ is a one-dimensional Gorenstein local ring with decreasing Hilbert function for any a and b. In particular we note that $R(I)_{-1,0}$ is isomorphic to the amalgamated duplication, that is not a domain, while $R(I)_{0,0}$ is isomorphic to the idealization (or trivial extension) that is never reduced.

THANK YOU!