Star operations on numerical semigroups

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Star operations

Let D be an integral domain with quotient field K, and let $\mathcal{F}(D) := \{I \subseteq K \mid xI \text{ is an ideal of } D \text{ for some } x \in K\}$ be the set of fractional ideals of D.

Definition

A star operation on D is a map $\star : \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathcal{F}(D)$, $x \in K$:

- *$\star$ is extensive: $I \subseteq I^\star$;*
- *$\star$ is idempotent: $(I^\star)^\star = I^\star$;*
- *$\star$ is order-preserving: if $I \subseteq J$, then $I^\star \subseteq J^\star$;*
- *$D^\star = D$;*
- *$x \cdot I^\star = (xI)^\star$.*

- Linked to the study of factorization, Krull domains, Kronecker function rings, integral closure of ideals, overrings of D...

Star operations on semigroups

Let S be a numerical semigroup.

- A fractional ideal of S is a subset $I \subseteq \mathbb{Z}$ such that $d + I$ is an ideal of S for some $d \in \mathbb{Z}$.
  - Equivalently, is a subset $I \subseteq \mathbb{Z}$ such that $I + S \subseteq I$ and $d + I \subseteq S$ for some $d \in \mathbb{Z}$.
  - We denote the set of fractional ideal of S as $\mathcal{F}(S)$.

Definition ([Kim, Kwak and Park 2001])

A star operation on S is a map $\star : \mathcal{F}(S) \longrightarrow \mathcal{F}(S)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathcal{F}(S)$, $x \in K$:

- *$\star$ is extensive: $I \subseteq I^\star$;*
- *$\star$ is idempotent: $(I^\star)^\star = I^\star$;*
- *$\star$ is order-preserving: if $I \subseteq J$, then $I^\star \subseteq J^\star$;*
- *$S^\star = S$;*
- *$d + I^\star = (d + I)^\star$.*

Numerical semigroup rings

- Given a field K, we can associate to S the integral domain

$$K[[S]] := K[[X^S]] = K[[\{X^s \mid s \in S\}]] = \left\{ \sum_{i \geq 0} a_i X^i \mid a_i = 0 \text{ if } i \notin S \right\}.$$

- $K[[S]]$ is a one-dimensional Noetherian local ring, and its integral closure is $K[[X]]$.
- There are many links between the structure of S and the structure of $K[[S]]$ [Barucci, Dobbs and Fontana 1997].
- Rings of the form $K[[S]]$, or similar rings, are used as examples in counting star operations [Houston, Mimouni and Park 2012].

Notation

Let S be a numerical semigroup.

- $\mathcal{F}(S)$ is the set of fractional ideals of S.
- $F(S) := \sup(\mathbb{Z} \setminus S)$ is the Frobenius number of S.
- $g(S) := |\mathbb{N} \setminus S|$ is the genus of S.
- $\mu(S) := \inf(S \setminus \{0\})$ is the multiplicity of S.
- $\mathrm{Star}(S)$ is the set of star operations on S.

Examples

- The identity $d : I \mapsto I$ is a star operation.
- If $\{S_\alpha \mid \alpha \in A\}$ are semigroups and $\bigcap_{\alpha\in A} S_\alpha = S$, then

$$I \mapsto \bigcap_{\alpha\in A} I + S_\alpha \quad \text{is a star operation.}$$

- The divisorial closure (or v-operation) is the map

$$v : J \mapsto J^v := (S - (S - J)).$$

  - Ideals that are v-closed are called *divisorial ideals*.
  - $\star_1 \leq \star_2$ if and only if $I^{\star_1} \subseteq I^{\star_2}$ for every I, or equivalently if every $\star_2$-closed ideal is $\star_1$-closed.
  - The v-operation is the biggest star operation; hence every divisorial ideal is $\star$-closed for every $\star \in \mathrm{Star}(S)$.
  - S and $\mathbb{N}$ are divisorial (over S).
  - $d = v$ (and so $|\mathrm{Star}(S)| = 1$) if and only if S is symmetric [Barucci, Dobbs and Fontana 1997].

Problems

- Given S, find a way to describe $\mathrm{Star}(S)$ (the maps, the cardinality, the order).
- Describe $\mathrm{Star}(S)$ for whole classes of semigroups.
- Find a formula to calculate the cardinality of $\mathrm{Star}(S)$ in a general way. At least, find estimates.
- Given n, which numerical semigroups have exactly n star operations?
- Extend the results to rings of the type $K[[S]]$.

Reduction to $\mathcal{F}_0(S)$

- By definition, if we know $I^\star$ we know also $(d+I)^\star$ for every $d \in \mathbb{Z}$.
- $\mathcal{F}_0(S)$ is the set of fractional ideals of S whose minimal element is 0.
  - Equivalently, is the set of fractional ideals I of S such that $S \subseteq I \subseteq \mathbb{N}$.
  - For every $I \in \mathcal{F}(S)$, there is a unique $d \in \mathbb{Z}$ such that $d + I \in \mathcal{F}_0(S)$.
  - Since $\mathbb{N} \setminus S$ is finite, so is $\mathcal{F}_0(S)$.
- Since S and $\mathbb{N}$ are divisorial, $\star$ restricts to a map $\star_0 : \mathcal{F}_0(S) \longrightarrow \mathcal{F}_0(S)$, and $\star_0$ uniquely determines $\star$.
- $\mathrm{Star}(S)$ is always finite.

Closed ideals

- If $I = I^\star$, I is said to be $\star$-*closed*.
- $\star$ is uniquely determined by the $\star$-closed ideals, since

$$I^\star = \bigcap\{J \mid I \subseteq J,\ J = J^\star\}.$$

- Moreover, $\star$ is uniquely determined by

$$\mathcal{F}_0^\star(S) := \{I \in \mathcal{F}_0(S) \mid I = I^\star\}.$$

- Let $\Delta \subseteq \mathcal{F}_0(S)$. Then, $\Delta = \mathcal{F}_0^\star(S)$ for some $\star \in \mathrm{Star}(S)$ if and only if:
  - $S \in \Delta$;
  - Δ is closed by intersections;
  - if $I \in \Delta$ and $k \in I$, then the k-shift $(-k + I) \cap \mathbb{N}$ is in Δ.
- These conditions can be checked in finite time.
- However, this algorithm is *very* slow.

Principal star operations

- We can attach to any fractional ideal I the star operation

$$\star_I : J \mapsto (S-(S-J)) \cap (I-(I-J)).$$

  - Equivalently, $\star_I$ is the biggest star operation closing I.
  - If $\star \in \mathrm{Star}(S)$, there are $I_1, \ldots, I_n$ such that $\star = \star_{I_1} \wedge \cdots \wedge \star_{I_n}$.
- If $I = I^v$, then $\star_I = v$.
- Let $\mathcal{G}_0(S) := \{I \in \mathcal{F}_0(S) \mid I \neq I^v\}$.
- If $I, J \in \mathcal{G}_0(S)$ and $I \neq J$ then $\star_I \neq \star_J$.
- $|\mathrm{Star}(S)| \geq |\mathcal{G}_0(S)| + 1$.

Generating nondivisorial ideals

- If S is not symmetric, there is λ such that $\lambda, F(S)-\lambda \notin S$; let $x \in \mathbb{N} \setminus S$.
  - If $x > \lambda$, define $I_x := \{y \in \mathbb{N} \mid x - y \notin S\}$.
  - If $x \leq \lambda$ and $\lambda - x \in S$, define $I_x := S \cup \{y \in \mathbb{N} \mid y > x\}$.
  - If $x \leq \lambda$ and $\lambda - x \notin S$, define $I_x := S \cup \{y \in \mathbb{N} \mid y > x, \lambda - y \notin S\}$.
- Every I_x is not divisorial, and they are all different from each other.
- $|\mathcal{G}_0(S)| \geq g(S)$.
- $|\mathrm{Star}(S)| \geq g(S) + 1$.
- For any g, there are only a finite number of numerical semigroups with $g(S) \leq g$.

Theorem

If $n > 1$, there are only a finite number of numerical semigroups with exactly n star operations.

An explicit version

Definition

- *$\xi(n)$ is the number of numerical semigroups with exactly n star operations.*
- *$\Xi(n)$ is the number of numerical semigroups S such that $2 \leq |\mathrm{Star}(S)| \leq n$.*
- *$\xi_\mu(n)$ and $\Xi_\mu(n)$ are as above, but restricted to semigroups of multiplicity μ.*

- Since we are doing estimates, it is more efficient to use $\Xi(n)$ than $\xi(n)$.
- [Zhai 2013] The number of numerical semigroups with $g(S) \leq g$ is asymptotic to $C\phi^g$ for some constant C, where ϕ is the golden ratio.
- $\Xi(n) = O(\phi^n) = O(\exp(n \log \phi))$.
- $\Xi_\mu(n) \leq \binom{n-1}{\mu-1} \leq (n-1)^{\mu-1}$.

Antichains

- Every $\Delta \subseteq \mathcal{G}_0(S)$ generates the star operation

$$J \mapsto \bigcap\{J^{\star_I} \mid I \in \Delta\} = (S-(S-J)) \cap \bigcap_{I\in\Delta}(J-(J-I)).$$

  - It can be $\star_\Delta = \star_\Lambda$ even if $\Delta \neq \Lambda$.
  - For example, if $J = J^{\star_I}$, then $\star_I = \star_{\{I,J\}}$.
- We say that $I \leq_\star J$ if I is $\star_J$-closed, i.e., is $\star_I \geq \star_J$ ($\star$-order).
- We consider star operations generated by *antichains* of $(\mathcal{G}_0(S), \leq_\star)$.
  - An antichain is a set of pairwise noncomparable elements.
  - This solves the problem of $J = J^{\star_I}$: $\{I, J\}$ is not an antichain.
  - However, different antichains can generate the same star operation.
- A more efficient algorithm: instead of all subsets of $\mathcal{F}_0(S)$, it is enough to consider sets of the form

$$\Delta^{\downarrow} := \{J \in \mathcal{F}_0(S) \mid J = J^v \text{ or } J \leq_\star I \text{ for some } I \in \Delta\},$$

where Δ is an antichain of $\mathcal{G}_0(S)$. Also, we only have to check intersections.

Atoms

- An atom of $\mathcal{G}_0(S)$ is an ideal I such that, if $I = I^{\star_1 \wedge \star_2}$, then $I = I^{\star_1} \cap I^{\star_2}$.
    - This means that, if $\star_I \geq \star_1 \wedge \star_2$, then $\star_I \geq \star_1$ or $\star_I \geq \star_2$ (a primality condition).
- If $\Delta \neq \Lambda$ are sets of atoms and are antichains in the $\star$-order, then $\star_\Delta \neq \star_\Lambda$.
- Not every ideal is an atom. Sufficient conditions:
    - $|I^v \setminus I| = 1$;
    - the set $\{I^\star \mid \star \in \mathrm{Star}(S)\}$ is linearly ordered.

The $\mathcal{Q}_a$

- Let $a \in \mathbb{N} \setminus S$. We consider the set

$$\mathcal{Q}_a := \{I \in \mathcal{G}_0(S) \mid \sup(\mathbb{N} \setminus I) = a,\ a \in I^v\}.$$

  - $\mathcal{Q}_a \neq \emptyset$ if $a \geq \lambda$, where $\lambda, F(S) - \lambda \notin S$.
  - In particular, if $a \geq F(S)/2$.
- Let $M_a := \{y \in \mathbb{N} \mid a - y \notin S\}$.
  - M_a is the biggest ideal of $\mathcal{Q}_a$, and its maximum in the $\star$-order.
  - M_a is an atom.
  - If $I \in \mathcal{Q}_a$ and $|M_a \setminus I| = 1$, then I is an atom.
- We can find antichain in $\mathcal{Q}_a$.
  - For ideals in $\mathcal{Q}_a$, every antichain with respect to containment is an antichain in the $\star$-order.
  - Better, every antichain with respect to containment generates a different star operation.
  - Even better, the same happens if we consider antichains in $\mathcal{Q}_a$ and $\mathcal{Q}_b$ for $a \neq b$ (so we can mix different kinds of constructions).

An example

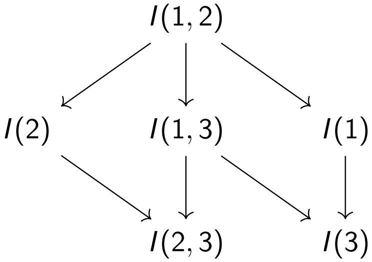
Let $S := \langle 4, 5, 6, 7\rangle = \{0, 4, \rightarrow\}$.

- There are eight elements in $\mathcal{F}_0(S)$:
  - S
  - $\mathbb{N}$
  - $I(1) := S \cup \{1\}$
  - $I(2) := S \cup \{2\}$
  - $I(3) := S \cup \{3\}$
  - $I(1, 2) := S \cup \{1, 2\}$
  - $I(1, 3) := S \cup \{1, 3\}$
  - $I(2, 3) := S \cup \{2, 3\}$

An example

Let $S := \langle 4, 5, 6, 7 \rangle = \{0, 4, \rightarrow\}$.

- There are eight elements in $\mathcal{F}_0(S)$:
  - $S = S^v$
  - $\mathbb{N} = \mathbb{N}^v$
  - $I(1) := S \cup \{1\} \in \mathcal{Q}_3$
  - $I(2) := S \cup \{2\} \in \mathcal{Q}_3$
  - $I(3) := S \cup \{3\} \in \mathcal{Q}_2$
  - $I(1,2) := S \cup \{1,2\} = M_3$
  - $I(1,3) := S \cup \{1,3\} = M_2$
  - $I(2,3) := S \cup \{2,3\} = M_1$
- Each ideal of $\mathcal{G}_0(S)$ is an atom.



$$|\mathrm{Star}(S)| = 14$$

A bound on multiplicity

- If $a, F(S) - a \notin S$, let $\mathcal{H} := \{x \in \mathbb{N} \setminus S \mid a - \mu(S) < x < a\}$.
  - If $\mu(S) < a \leq F(S)/2$, then $|\mathcal{H}| \geq \lfloor \mu(S)/2 \rfloor$.
  - Let $I := S \cup \{x \in \mathbb{N} \mid x > a\}$.
  - If $H \subseteq \mathcal{H}$, then $I \cup H$ is an ideal in $\mathcal{Q}_a$.
  - Every family of noncomparable subsets of $\mathcal{H}$ generates a different star operation.
- $$|\mathrm{Star}(S)| \geq \exp\left(\binom{\lfloor \mu/2 \rfloor}{\lfloor \mu/4 \rfloor} \log(2)\right).$$
- Writing $\Xi(n) = \sum_\mu \Xi_\mu(n)$, we obtain

$$\Xi(n) = O(n^{(A+\epsilon)\log\log(n)})$$

for every $\epsilon > 0$, where $A := \frac{2}{\log(2)}$.

Multiplicity 3

- Let $S := \langle 3, 3\alpha+1, 3\beta+2\rangle$. Then, $\mathcal{G}_0(S)$ is order-isomorphic to a rectangle with sides of length $2\alpha-\beta$ and $2\beta-\alpha+1$.
- From this, we can calculate

$$|\mathrm{Star}(S)| = \binom{\alpha+\beta+1}{2\alpha-\beta} = \binom{\alpha+\beta+1}{2\beta-\alpha+1} = \binom{g(S)+1}{F(S)-g(S)+2}.$$

- The numerical semigroups of multiplicity 3 with n star operations corresponds to the binomial coefficients $\binom{x}{y}$ such that $x+y \equiv 1 \bmod 3$. Hence,

$$\xi_3(n) = O(\log(n)) \quad \text{and} \quad \Xi_3(n) = \frac{2}{3}n + O(\sqrt{n}\log(n)).$$

- If every integer is only equal to a finite number of binomial coefficients (a conjecture of Erdős), then the logarithms can be eliminated.

Other cases

- If S is pseudosymmetric and $F(S) = 2\mu(S) - 2$, then $|\mathrm{Star}(S)| = 1 + \omega(\mu - 2)$ (where $\omega(x)$ is the number of antichain of the power set of $\{1, \dots, x\}$).
- If S is pseudosymmetric and $\mu(S) = 4$, then $|\mathrm{Star}(S)| = 2^{\frac{g(S)}{2}+1} - 1$.
- Let $S_{j,k} := \langle 4, 4j + 2, 2k + 1, 2k + 4j - 1 \rangle$. Experimentally, we have

$$\begin{array}{rlrcl} |\mathrm{Star}(S_{1,k})| = & 20k-29 & \text{for } 4 & \leq k \leq & 13 \\ |\mathrm{Star}(S_{2,k})| = & 400k-1432 & \text{for } 7 & \leq k \leq & 15 \\ |\mathrm{Star}(S_{3,k})| = & 6800k-38200 & \text{for } 10 & \leq k \leq & 14 \end{array}$$

Some data

- $\xi(2) = 0$, $\xi(3) = 1$, $\xi(4) = 1$, $\xi(5) = 0$, $\xi(6) = 1$, $\xi(7) = 2$, $\xi(8) = 0$, $\xi(9) = 1$, $\xi(10) = 2$, $\xi(11) = 0$, $\xi(12) = 1$, $\xi(13) = 1$, $\xi(14) = 2$, $\xi(15) = 3$, $\xi(16) = 1$, $\xi(17) = 0$, $\dots$
- There are 43 numerical semigroups with 45 or less star operations.
  - 34 of these have multiplicity 3, 6 have multiplicity 4 and 3 have multiplicity 5.
  - 34 of these are pseudosymmetric.
  - 29 are pseudosymmetric of multiplicity 3.

Semigroup rings

- Every star operation on S induces a star operation on $K[[S]]$.
- Conversely, there are two canonical surjective maps from $\mathrm{Star}(K[[S]])$ to $\mathrm{Star}(S)$.
- $|\mathrm{Star}(K[[S]])| \geq |\mathrm{Star}(S)|$.
- $|\mathrm{Star}(K[[S]])| = 1$ if and only if $|\mathrm{Star}(S)| = 1$.
- For a fixed field K and a fixed $n > 1$, there are only finitely many rings of the form $K[[S]]$ with exactly n star operations.

Residually rational rings

- More generally: take a discrete valuation ring V, with valuation $\mathbf{v}$. Let $\mathfrak{V}(V)$ be the set of rings R such that:
  - the integral closure of R is V;
  - R is Noetherian;
  - $(R : V) \neq (0)$;
  - the inclusion $R \hookrightarrow V$ induces an isomorphism $R/\mathfrak{m}_R \xrightarrow{\simeq} V/\mathfrak{m}_V$.
- Every $R \in \mathfrak{V}(V)$ is associated to the numerical semigroup $\mathbf{v}(R)$.
- We can't apply directly the semigroup case: R has more ideals than $\mathbf{v}(R)$, but some ideals of $\mathbf{v}(R)$ does not correspond to ideals of R.
- However, we can replay the arguments of the semigroup case.
- If the residue field of V is finite and $n > 1$, then there are only finitely many $R \in \mathfrak{V}(V)$ such that $|\mathrm{Star}(R)| = n$.

Thank you for your attention

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