0-th local cohomology of tangent cones of monomial space curves

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Introduction

Let $S = \langle g_1, g_2, g_3 \rangle \subseteq \mathbb{N}$ be a numerical semigroup with $g_1 < g_2 < g_3$ and

$$R = \Bbbk[[t^{g_1}, t^{g_2}, t^{g_3}]] \subseteq \Bbbk[[t]]$$

the corresponding numerical semigroup ring, where $\Bbbk$ is a field.

The associated graded ring is $G := \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$.

It is important to understand when G is a Cohen-Macaulay ring, that is $\dim G = \operatorname{depth} G$; in this case, this is equivalent to $(t^{g_1})^*$ being a non-zerodivisor.

Failure of the Cohen-Macaulay property of G

The 0-th local cohomology module of G with respect to the homogeneous maximal ideal $\mathfrak{m} \subseteq G$ is

$$H^0_{\mathfrak{m}}(G) := \bigcup_{i=1}^{\infty} \left(0 :_G \mathfrak{m}^i\right) = \bigcup_{i=1}^{\infty} \left(0 :_G ((t^{g_1})^*)^i\right).$$

By depth sensitivity we have

$$G \text{ is Cohen-Macaulay} \iff H^0_{\mathfrak{m}}(G) = 0.$$

$H^0_{\mathfrak{m}}(G)$ is a monomial ideal of G such that

- $\ell(H^0_{\mathfrak{m}}(G)) < \infty$;
- $\mathfrak{m}^k H^0_{\mathfrak{m}}(G) = 0$ for some $k \geq 0$.

The Buchsbaum property

Definition

G is called a Buchsbaum ring if $\mathfrak{m}H^0_{\mathfrak{m}}(G) = 0$.

Conjecture (Sapko, 2001)

Let $S = \langle g_1, g_2, g_3 \rangle$. The associated graded ring $G = \oplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is Buchsbaum if and only if $\ell(H^0_{\mathfrak{m}}(G)) \leq 1$.

The conjecture was proved independently by Shen and by D'Anna-Micale-S. in 2011.

Proof (Cortadellas-Jafari-Zarzuela, 2013).

If S is 3-generated, then $H^0_{\mathfrak{m}}(G)$ is a principal ideal generated by an element of the form $((t^{g_3})^*)^i$. □

The 2-Buchsbaum case

Definition

G is called a k-Buchsbaum ring if $\mathfrak{m}^k H^0_{\mathfrak{m}}(G) = 0$.

Thus:

$$\begin{aligned} \text{0-Buchsbaum} &= \text{Cohen-Macaulay property} \\ \text{1-Buchsbaum} &= \text{Buchsbaum property} \end{aligned}$$

Theorem (Shen, 2011)

Assume that $S = \langle g_1, g_2, g_3 \rangle$. The associated graded ring $G = \oplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is 2-Buchsbaum if and only if $\ell(H^0_{\mathfrak{m}}(G)) \leq 2$.

Generalization for k-Buchsbaum associated graded rings

If $\ell(H^0_{\mathfrak{m}}(G)) \leq k$ for some $k \in \mathbb{N}$ then G is k-Buchsbaum.

For $k = 0, 1, 2$ the converse holds.
However, Cortadellas-Jafari-Zarzuela (2013) observe that this fails in general.

Example

Let $S = \langle 6, 7, 16 \rangle$. Then G is 3-Buchsbaum but $\ell(H^0_{\mathfrak{m}}(G)) = 4$.

Still, they show that $\sup\{\ell(H^0_{\mathfrak{m}}(G)) : G \text{ is } k\text{-Buchsbaum}\} < \infty$ for every k.

Question

Assume that $S = \langle g_1, g_2, g_3 \rangle$. If the associated graded ring G is k-Buchsbaum, what is the largest possible value of $\ell(H^0_{\mathfrak{m}}(G))$?

A conjectural answer

Conjecture

Let $S = \langle g_1, g_2, g_3 \rangle$. If G is a k-Buchsbaum ring, i.e. $\mathfrak{m}^k H^0_\mathfrak{m}(G) = 0$, then

$$\ell(H^0_\mathfrak{m}(G)) \leq \left\lfloor \frac{k+2}{3} \right\rfloor \left\lfloor \frac{k+3}{3} \right\rfloor \left\lfloor \frac{k+4}{3} \right\rfloor$$

and this bound is sharp for each k.

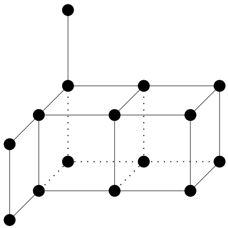
Computational evidence: true for $g_1, g_2, g_3 \leq 300$.

Combinatorial interpretation

Recall: $R = \Bbbk[[t^{g_1}, t^{g_2}, t^{g_3}]]$, $G = \oplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1} = \Bbbk[x, y, z]$ where $x = (t^{g_1})^*$, $y = (t^{g_2})^*$, $z = (t^{g_3})^*$.

We associate to each monomial $\mathbf{u} \in H^0_{\mathfrak{m}}(G)$ its factorization $\mathbf{u} = x^a y^b z^c$:

$$\mathcal{H} := \left\{ (a, b, c) \,|\, 0 \neq x^a y^b z^c \in H^0_{\mathfrak{m}}(G) \right\} \subseteq \mathbb{N}^3$$



$\ell(H^0_{\mathfrak{m}}(G))$	$\leftrightarrow$	$\mathrm{vol}(\mathcal{H})$
k-Buchsbaum property	$\leftrightarrow$	number of slices of $\mathcal{H}$

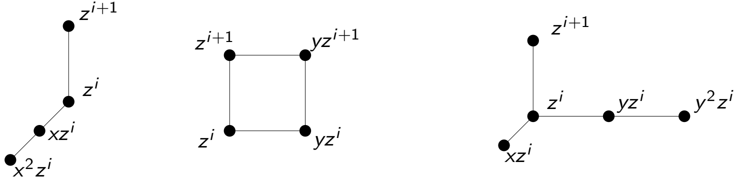
Conjecture (revisited)

The volume of $\mathcal{H}$ is less than or equal to the largest possible volume of a rectangular parallelepiped with as many slices as $\mathcal{H}$.

Examples

Recall: $x = (t^{g_1})^*$, $y = (t^{g_2})^*$, $z = (t^{g_3})^*$.

Structure of $H^0_{\mathfrak{m}}(G) = z^i G$ when G is 3-Buchsbaum (3 slices)



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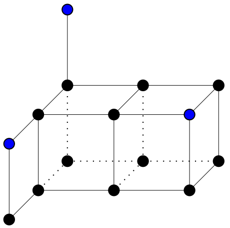
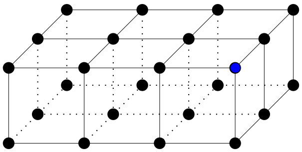
According to the conjecture $\ell(H^0_{\mathfrak{m}}(G)) \leq 4$.

The socle (1)

Since $H^0_{\mathfrak{m}}(G) = z^i G$ for some $i \geq 0$, the crucial part is to understand the socle

$$\mathrm{Soc}(G) := (0 :_G \mathfrak{m}) \subseteq H^0_{\mathfrak{m}}(G) = \cup_i (0 :_G \mathfrak{m}^i).$$





Remark

Suppose $\dim_{\Bbbk} \mathrm{Soc}(G) = 1$. If G is k-Buchsbaum then

$$\ell(H^0_{\mathfrak{m}}(G)) \leq \left\lfloor \frac{k+2}{3} \right\rfloor \left\lfloor \frac{k+3}{3} \right\rfloor \left\lfloor \frac{k+4}{3} \right\rfloor.$$

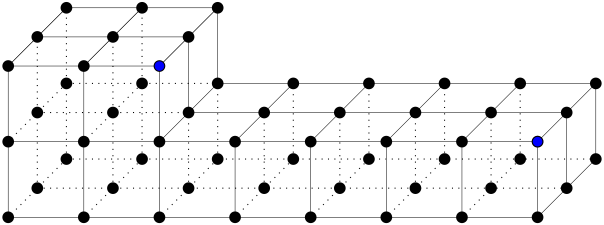
The socle (2)

Unfortunately, $\dim_{\Bbbk} \mathrm{Soc}(G)$ can be large.

Proposition

Suppose that $S = \langle g_1, g_2, g_3 \rangle$ is symmetric. If all the elements of $\{(a,b,c) \,|\, x^a y^b z^c \in \mathrm{Soc}(G)\} \subseteq \mathbb{N}^3$ have a constant coordinate, then

$$\ell(H^0_{\mathfrak{m}}(G)) \leq \left\lfloor \frac{k+2}{3} \right\rfloor \left\lfloor \frac{k+3}{3} \right\rfloor \left\lfloor \frac{k+4}{3} \right\rfloor .$$



Sharpness

Theorem

For every $k \in \mathbb{N}$ there exists a 3-generated numerical seimigroup ring $(R, \mathfrak{m})$ such that the associated graded ring G is k-Buchsbaum and

$$\ell(H^0_{\mathfrak{m}}(G)) = \left\lfloor \frac{k+2}{3} \right\rfloor \left\lfloor \frac{k+3}{3} \right\rfloor \left\lfloor \frac{k+4}{3} \right\rfloor .$$

Proof.

Use the "six parameters" from Rosales-García-Sánchez (2004) to construct families with prescribed socle.

$$S_k = \begin{cases} \langle 3p^2+4p, 3p^2+5p, 3p^2+12p+11 \rangle & \text{if } k = 3p; \\ \langle 3p^2+9p+6, 3p^2+9p+7, 3p^2+12p+11 \rangle & \text{if } k = 3p+1; \\ \langle 3p^2+10p+8, 3p^2+10p+9, 3p^2+13p+14 \rangle & \text{if } k = 3p+2. \end{cases}$$

□