Pseudo symmetric monomial curves

Mesut Şahin

HACETTEPE UNIVERSITY

Joint work with Nil Şahin (Bilkent University)

Supported by Tubitak No: 114F094

IMNS 2016, July 4-8, 2016

Part I

Indispensability

Semigroup, toric ideal, semigroup ring

Let $n_1, \dots, n_4$ be positive integers with $\gcd(n_1, \dots, n_4) = 1$. Then

$S = \langle n_1, \dots, n_4 \rangle$ is $\{u_1 n_1 + \cdots + u_4 n_4 \mid u_i \in \mathbb{N}\}$. Let K be a field and

$K[S] = K[t^{n_1}, \dots, t^{n_4}]$ be the semigroup ring of S, then $K[S] \simeq A/I_S$

where, $A = K[X_1, \dots, X_4]$ and the toric ideal I_S is the kernel of the

surjection $A \xrightarrow{\phi_0} K[S]$, where $X_i \mapsto t^{n_i}$.

Pseudo symmetric S

Pseudo frobenious numbers of S are defined to be the elements of the set

$PF(S) = \{n \in \mathbb{Z} - S \mid n + s \in S \text{ for all } s \in S - \{0\}\}$. The largest element

is called the frobenious number denoted by $g(S)$.

S is called pseudo symmetric if $PF(S) = \{g(S)/2, g(S)\}$.

S is symmetric if $PF(S) = \{g(S)\}$.

S is pseudo symmetric if $PF(S) = \{g(S)/2, g(S)\}$.

Recall the set $PF(S) = \{n \in \mathbb{Z} - S \mid n + s \in S \text{ for all } s \in S - \{0\}\}$.

$S = \langle 5, 12, 11, 14 \rangle = \{0, 5, 10, 11, 12, 14, 15, 16, 17, 19\} + \mathbb{N}$ and its

complement is $\{1, 2, 3, 4, 6, 7, 8, 9, 13, 18\}$.

$1 + 5, 2 + 5, 3 + 5, 4 + 5, 6 + 12, 7 + 11, 8 + 10, 13 + 5 \notin S$ but

$n + s \in S$ for all $s \in S - \{0\}$, for $n = 9, 18$. So, S is pseudosymmetric.

Komeda proved that, the semigroup S is pseudo symmetric if and only if

there are positive integers α_i, $1 \leq i \leq 4$, and α_{21}, with $\alpha_{21} < \alpha_1$, s.t.

$n_1 = \alpha_2\alpha_3(\alpha_4 - 1) + 1,$

$n_2 = \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3,$

$n_3 = \alpha_1\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1,$

$n_4 = \alpha_1\alpha_2(\alpha_3 - 1) + \alpha_{21}(\alpha_2 - 1) + \alpha_2.$

For $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{21}) = (5, 2, 2, 2, 2)$, $S = \langle 5, 12, 11, 14 \rangle$.

Pseudo symmetric S

Komeda proved that, $K[S] = A/(f_1, f_2, f_3, f_4, f_5)$, where

$$f_1 = X_1^{\alpha_1} - X_3X_4^{\alpha_4-1},$$

$$f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}}X_4,$$

$$f_3 = X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2,$$

$$f_4 = X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1},$$

$$f_5 = X_3^{\alpha_3-1}X_1^{\alpha_{21}+1} - X_2X_4^{\alpha_4-1}.$$

S-degrees

Let $\deg_S(X_1^{u_1}X_2^{u_2}X_3^{u_3}X_4^{u_4}) = \sum_{i=1}^{4} u_i n_i \in S$. $d \in S$ is called a Betti S-degree if there is a minimal generator of I_S of S-degree d and β_d is the number of times d occurs as a Betti S-degree. Both β_d and the set B_S of Betti S-degrees are invariants of I_S. S-degrees of binomials in I_S which are not comparable with respect to $<_S$ constitute the minimal binomial S-degrees denoted M_S, where $s_1 <_S s_2$ if $s_2 - s_1 \in S$. In general, $M_S \subseteq B_S$.

Indispensables

By Komeda's result, $B_S = \{d_1, d_2, d_3, d_4, d_5\}$ if d_i's are all distinct, where d_i is the S-degree of f_i, for $i = 1, \ldots, 5$. A binomial is called indispensable if it appears in every minimal generating set of I_S.

Lemma

A binomial of S-degree d is indispensable if and only if $\beta_d = 1$ and $d \in M_S$.

We use the following Lemma twice in the sequel.

If $0 < v_k < \alpha_k$ and $0 < v_l < \alpha_l$, for $k \neq l \in \{1, 2, 3, 4\}$, then

$v_k n_k - v_l n_l \notin S$.

Proposition

$M_S = \{d_1, d_2, d_3, d_4, d_5\}$ if $\alpha_1 - \alpha_{21} > 2$ and $M_S = \{d_1, d_2, d_3, d_5\}$ if

$\alpha_1 - \alpha_{21} = 2$.

Corollary

Indispensable binomials of I_S are $\{f_1, f_2, f_3, f_4, f_5\}$ if $\alpha_1 - \alpha_{21} > 2$ and are

$\{f_1, f_2, f_3, f_5\}$ if $\alpha_1 - \alpha_{21} = 2$.

Strongly indispensable minimal free resolutions

For a graded minimal free A-resolution

$$\mathbf{F}: \ 0 \longrightarrow A^{\beta_{k-1}} \xrightarrow{\phi_{k-1}} A^{\beta_{k-2}} \xrightarrow{\phi_{k-2}} \cdots \xrightarrow{\phi_2} A^{\beta_1} \xrightarrow{\phi_1} A^{\beta_0} \longrightarrow K[S] \longrightarrow 0$$

of $K[S]$, let A^{β_i} be generated in degrees $s_{i,j} \in S$, which we call i-Betti degrees, i.e. $A^{\beta_i} = \bigoplus_{j=1}^{\beta_i} A[-s_{i,j}]$.

The resolution $(\mathbf{F}, \phi)$ is strongly indispensable if for any graded minimal resolution $(\mathbf{G}, \theta)$, we have an injective complex map $i: (\mathbf{F}, \phi) \longrightarrow (\mathbf{G}, \theta)$.

Betti i-degrees of $S = \langle 5, 12, 11, 14 \rangle$

$1-$Betti degrees : $\{22, 24, 25, 26, 28\}$

$2-$Betti degrees: $\{36, 37, 38, 39, 40, 46\}$

$3-$Betti degrees : $\{51, 60\}$.

Note that $\{51, 60\} - 42 = \{9, 18\}$. Recall that

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{21}) = (5, 2, 2, 2, 2)$.

In, Barucci-Froberg-Sahin (2014), we give a minimal free resolution of $K[S]$, for symmetric and pseudo symmetric S and prove that it is always strongly indispensable for symmetric S. It follows that it is strongly indispensable for pseudo symmetric S iff the differences between the $i-$Betti degrees do not lie in S, for only $i = 1, 2$. Using this, we obtain

Main Theorem 1

Let S be a 4-generated pseudo-symmetric semigroup. Then $K[S]$ has a strongly indispensable minimal graded free resolution if and only if $\alpha_4 > 2$ and $\alpha_1 - \alpha_{21} > 2$.

Part II

Cohen-Macaulayness of the Tangent Cone

and Sally's Conjecture

If $(R, \mathbf{m})$ is a local ring with maximal ideal $\mathbf{m}$, then the Hilbert function of R is defined to be the Hilbert function of its *associated graded ring*

$$gr_{\mathbf{m}}(R) = \bigoplus_{r \in \mathbb{N}} \mathbf{m}^r / \mathbf{m}^{r+1}.$$

That is,

$$H_R(r) = dim_K(\mathbf{m}^r / \mathbf{m}^{r+1}).$$

The Main Problem:

Determine the conditions under which the Hilbert function of a local ring $(R, \mathbf{m})$ is non-decreasing.

A sufficient condition:

If the tangent cone is Cohen-Macaulay, $H_R(r)$ is non-decreasing. But this does not follow from Cohen-Macaulayness of $(R, \mathbf{m})$.

Sally's Conjecture (1980):

If $(R, \mathbf{m})$ is a one dimensional Cohen-Macaulay local ring with small embedding dimension $d := H_R(1)$, then $H_R(r)$ is non-decreasing.

Literature:

- $d = 1$, obvious as $H_R(r) = 1$
- $d = 2$, proved by Matlis (1977)
- $d = 3$, proved by Elias (1993)
- $d = 4$, a counterexample is given by Gupta-Roberts (1983)
- $d \geq 5$, counterexamples for each d are given by Orecchia(1980).

The local ring associated to the monomial curve $C = C(n_1, \dots, n_k)$ is $K[[t^{n_1}, \dots, t^{n_k}]]$ with $\mathbf{m} = (t^{n_1}, \dots, t^{n_k})$, and the associated graded ring $gr_{\mathbf{m}}(K[[t^{n_1}, \dots, t^{n_k}]])$ is isomorphic to the ring $K[x_1, \dots, x_k]/I(C)_*$, where $I(C)$ is the defining ideal of C and $I(C)_*$ is the ideal generated by the polynomials f_* for f in $I(C)$ and f_* is the homogeneous summand of f of least degree. In other words, $I(C)_*$ is the defining ideal of the tangent cone of C at 0.

Herzog-Waldi,1975

Let $C = C(30, 35, 42, 47, 148, 153, 157, 169, 181, 193) \subset \mathbb{A}^{10}$ and $(R, \mathbf{m})$ be its associated local ring. Then the Hilbert function of R is NOT non-decreasing as $H_R = \{1, 10, \mathbf{9}, 16, 25, \ldots\}$.

Eakin-Sathaye,1976

Let $C = C(15, 21, 23, 47, 48, 49, 50, 52, 54, 55, 56, 58) \subset \mathbb{A}^{12}$ and $(R, \mathbf{m})$ be its associated local ring. Then the Hilbert function of R is NOT non-decreasing as $H_R = \{1, 12, \mathbf{11}, 13, 15, \ldots\}$.

4-generated case:

The conjecture has been proven by Arslan-Mete in 2007 for Gorenstein local rings R associated to certain symmetric monomial curves in $\mathbb{A}^4$. The method to achieve this result was to show that the tangent cones of these curves at the origin are Cohen-Macaulay. More recently, Arslan-Katsabekis-Nalbandiyan, generalized this characterizing Cohen-Macaulayness of the tangent cone completely.

Criterion for Cohen-Macaulayness

Let $C = C(n_1, \dots, n_k)$ be a monomial curve with n_1 the smallest and $G = \{f_1, \dots, f_s\}$ be a minimal standard basis of the ideal $I(C)$ wrt the negative degree reverse lexicographical ordering that makes x_1 the lowest variable. C has Cohen-Macaulay tangent cone at the origin if and only if x_1 does not divide $\mathrm{LM}(f_i)$ for $1 \leq i \leq k$, where $\mathrm{LM}(f_i)$ denotes the leading monomial of a polynomial f_i.

Pseudo symmetric S with n_1 the smallest generator

Lemma

The set $G = \{f_1, f_2, f_3, f_4, f_5\}$ where f_i's are as defined before, is a minimal standard basis for I_S wrt negdegrevlex ordering, if

1. $\alpha_2 \leq \alpha_{21} + 1$
2. $\alpha_{21} + \alpha_3 \leq \alpha_1$
3. $\alpha_4 \leq \alpha_2 + \alpha_3 - 1.$

Pseudo symmetric S with n_1 the smallest generator

Main Theorem 2

Tangent cone of the monomial curve C_S is Cohen-Macaulay iff

1. $\alpha_2 \leq \alpha_{21} + 1$
2. $\alpha_{21} + \alpha_3 \leq \alpha_1$
3. $\alpha_4 \leq \alpha_2 + \alpha_3 - 1.$

Pseudo symmetric S with n_2 the smallest generator

Lemma

1. $\alpha_{21} + \alpha_3 \leq \alpha_1$
2. $\alpha_{21} + \alpha_3 \leq \alpha_4$
3. $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$
4. $\alpha_{21} + \alpha_1 \leq \alpha_4 + \alpha_2 - 1$ then a minimal standard basis for I_S is

(i) $\{f_1, f_2, f_3, f_4, f_5\}$ if $\alpha_1 \leq \alpha_4$,

(ii) $\{f_1, f_2, f_3, f_4, f_5, f_6 = X_1^{\alpha_1+\alpha_{21}} - X_2^{\alpha_2} X_3 X_4^{\alpha_4-2}\}$ if $\alpha_1 > \alpha_4$,

Pseudo symmetric S with n_2 the smallest generator

The Main Theorem 3

Tangent cone of the monomial curve C_S is Cohen-Macaulay iff

1. $\alpha_{21} + \alpha_3 \leq \alpha_1$
2. $\alpha_{21} + \alpha_3 \leq \alpha_4$
3. $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$
4. $\alpha_{21} + \alpha_1 \leq \alpha_4 + \alpha_2 - 1.$

Pseudo symmetric S with n_3 the smallest generator

Lemma

If the tangent cone of C_S is Cohen-Macaulay, then the following must hold

1. $\alpha_1 \leq \alpha_4$,
2. $\alpha_4 \leq \alpha_{21} + \alpha_3$,
3. $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$ if $\alpha_1 - \alpha_{21} > 2$; $\alpha_4 \leq \alpha_2 + 2\alpha_3 - 3$ if $\alpha_1 - \alpha_{21} = 2$,

Pseudo symmetric S with n_3 the smallest generator

Lemma

1. $\alpha_1 \leq \alpha_4$,
2. $\alpha_4 \leq \alpha_{21} + \alpha_3$,
3. $\alpha_2 \leq \alpha_{21} + 1$, then a minimal standard basis for I_S is

(i) $\{f_1, f_2, f_3, f_4, f_5, f_6 = X_1^{\alpha_1 - 1} X_4 - X_2^{\alpha_2 - 1} X_3^{\alpha_3}\}$ if $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$,

(ii) $\{f_1, f_2, f_3, f_4', f_5, f_6\}$ if $\alpha_1 - \alpha_{21} = 2$, $\alpha_2 + \alpha_3 - 1 < \alpha_4 \leq \alpha_2 + 2\alpha_3 - 3$.

Corollary

If n_3 is the smallest and

1. $\alpha_1 \leq \alpha_4$,
2. $\alpha_4 \leq \alpha_{21} + \alpha_3$,
3. $\alpha_2 \leq \alpha_{21} + 1$,
4. $\alpha_4 \leq \alpha_2 + \alpha_3 - 1$,

hold, then the tangent cone of the monomial curve C_S is Cohen-Macaulay.

If (1), (2), (3) hold, $\alpha_1 - \alpha_{21} = 2$ and $\alpha_2 + \alpha_3 - 1 < \alpha_4 \leq \alpha_2 + 2\alpha_3 - 3$, the tangent cone of C_S is Cohen-Macaulay if and only if $\alpha_1 \leq \alpha_2 + \alpha_3 - 1$.

Pseudo symmetric S with n_4 the smallest generator

Lemma

If the tangent cone of the monomial curve C_S is Cohen-Macaulay then

1. $\alpha_1 \leq \alpha_4$,
2. $\alpha_2 \leq \alpha_{21} + 1$,
3. $\alpha_3 + \alpha_{21} \leq \alpha_4$.

Lemma

Let n_4 be the smallest in $\{n_1, n_2, n_3, n_4\}$ and the conditions

1. $\alpha_1 \leq \alpha_4$,
2. $\alpha_2 \leq \alpha_{21} + 1$,
3. $\alpha_3 + \alpha_{21} \leq \alpha_4$,
4. $\alpha_3 \leq \alpha_1 - \alpha_{21}$,

hold, then $\{f_1, f_2, f_3, f_4, f_5\}$ is a minimal standard basis for I_S and the tangent cone of C_S is Cohen-Macaulay.

FINAL RESULT

Hilbert function of the local ring is non-decreasing, when the TC is

Cohen-Macaulay.

FINAL RESULT

Hilbert function of the local ring is non-decreasing, when the TC is

Cohen-Macaulay.

THANK YOU