# Common behaviours in families of numerical semigroups: types of Frobenius varieties

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# Trees

- A *graph* $\mathrm{G}$ is a pair $(V, E)$ where $V$ is a non-empty set (of *vertices*) and $E$ is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$ (the *edges* of $\mathrm{G}$).
- A *path (of length n)* connecting two vertices $x$ and $y$ is a sequence of different edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.
- A graph $\mathrm{G}$ is a *tree* if there exists a vertex $v^*$ (the *root* of G) such that, for every other vertex $x$, there exists a unique path connecting $x$ and $v^*$.
- If $(x, y)$ is an edge, then we say that $x$ is a child of $y$.

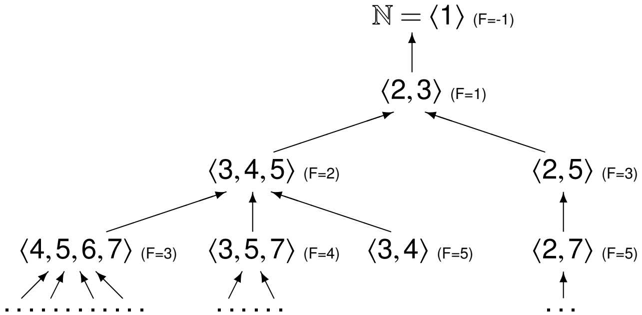
# The tree of the set of numerical semigroups

- Let $\mathcal{S}$ be the set formed by all numerical semigroups.
- Let $\mathrm{G}(\mathcal{S})$ be the tree associated to $\mathcal{S}$. We have that
  - the vertices are the elements of $\mathcal{S}$,
  - $(T,S)$ is an edge if $S = T \cup \{\mathrm{F}(T)\}$,
  - $\mathbb{N}$ is the root.
- If $S$ is a numerical semigroup, then the unique path connecting $S$ with $\mathbb{N}$ is given by the *chain of numerical semigroups associated to S*:
  - $\mathcal{C}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$,
    with $S_0 = S$ and $S_{i+1} = S_i \cup \{\mathrm{F}(S_i)\}$, for all $i < n$, and $S_n = \mathbb{N}$.
- The children of $S \in \mathcal{S}$ are $S \setminus \{a_1\}, \ldots, S \setminus \{a_r\}$, where $a_1, \ldots, a_r$ are the elements of $\mathrm{msg}(S)$ which are greater than $\mathrm{F}(S)$.

$*$ $\mathrm{msg}(S)$ is minimal system of generators of $S$.

# The tree of the set of numerical semigroups

- The first levels (with respect the genus) of $\mathrm{G}(\mathcal{S})$.




$*$ $\langle 3,4 \rangle$ is a *leaf*: it has not got any child.

# Purpose and tools

- Purpose: define structures (that is, the varieties) that allow us to build and to arrange the elements of families of numerical semigroups.
- Tools:
  - definitions of (several types of) variety,
  - monoid associated to a variety,
  - minimal system of generators with respect to a variety.

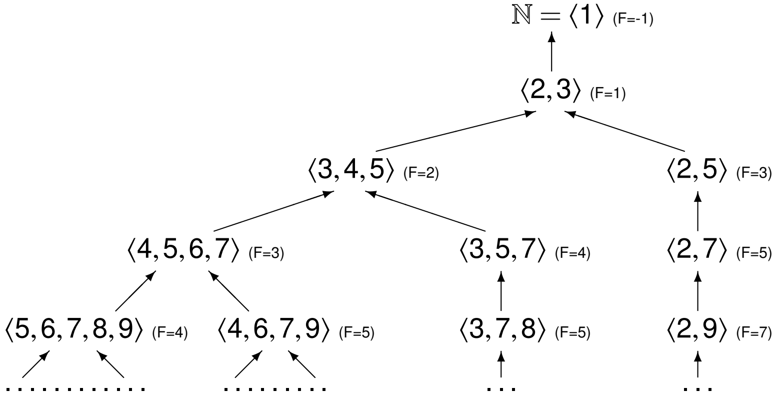
# Analysis of the set of numerical semigroups

- Let $S$ and $T$ be numerical semigroups (with $S \neq \mathbb{N}$).
  - $S \cup \{\mathrm{F}(S)\}$ and $S \cap T$ are numerical semigroups.
- Let $S$ be a numerical semigroup.
  - $S \setminus \{a\}$ is a numerical semigroup if and only if $a \in \mathrm{msg}(S)$.
- Let $S, T$ be numerical semigroups.
  - $S = T \cup \{\mathrm{F}(T)\}$ if and only if $T = S \setminus \{a\}$ for some $a \in \mathrm{msg}(S)$ such that $a > \mathrm{F}(S)$.
- Let $S, T$ be numerical semigroups such that $S = T \cup \{\mathrm{F}(T)\}$.
  - $\mathrm{F}(S) < \mathrm{F}(T)$ and $\mathrm{g}(T) = \mathrm{g}(S) + 1$.

# Frobenius varieties (Rosales, 2008)

- A *variety* is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfills the following conditions,
  - if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$;
  - if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.
- Examples of Frobenius varieties
  - Arf numerical semigroups.
  - Saturated numerical semigroups.
  - Numerical semigroups having a Toms decomposition.
  - Numerical semigroups defined by strongly admissible linear patterns.

# The tree of the Arf numerical semigroups




$*$ Binary tree: each node has at most two children.

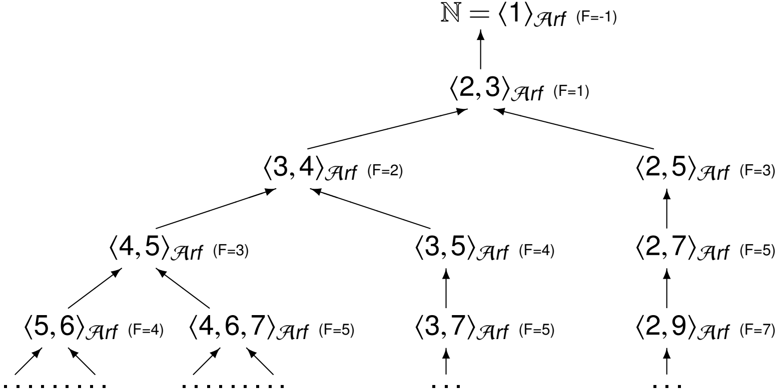
# $\mathcal{V}$-monoids and $\mathcal{V}$-systems of generators

- Let $\mathcal{V}$ be a Frobenius variety.
- A submonoid $M$ of $\mathbb{N}$ is a $\mathcal{V}$-monoid if it can be expressed as intersection of elements in $\mathcal{V}$.
  - The intersection of $\mathcal{V}$-monoids is a $\mathcal{V}$-monoid.
- Let $A \subseteq \mathbb{N}$. The *$\mathcal{V}$-monoid generated by $A$* (denoted by $\mathcal{V}(A)$) is the intersection of all the $\mathcal{V}$-monoids containing $A$.
  - $\mathcal{V}(A)$ is the intersection of all elements of $\mathcal{V}$ containing $A$.
- If $M = \mathcal{V}(A)$, then $A$ is a *$\mathcal{V}$-system of generators* of $M$.
- $A$ is a *minimal $\mathcal{V}$-system of generators* of $M$ if $M \neq \mathcal{V}(B)$ for all $B \subsetneq A$.
  - Every $\mathcal{V}$-monoid $M$ has a unique minimal $\mathcal{V}$-system of generators, which in additon is finite ($\mathrm{msg}_{\mathcal{V}}(M)$).
- If $M$ is a $\mathcal{V}$-monoid and $x \in M$, then $M \setminus \{x\}$ is a $\mathcal{V}$-monoid if and only if $x \in \mathrm{msg}_{\mathcal{V}}(M)$.

# The tree of a Frobenius variety

- Let $\mathcal{V}$ be a variety.
- Let $\mathrm{G}(\mathcal{V})$ be the tree associated to $\mathcal{V}$. We have that
  - the vertices are the elements of $\mathcal{V}$,
  - $(T, S)$ is an edge if $S = T \cup \{\mathrm{F}(T)\}$,
  - $\mathbb{N}$ is the root.
- If $S \in \mathcal{V}$, then the unique path connecting $S$ with $\mathbb{N}$ is $\mathcal{C}(S)$ (that is, the chain of numerical semigroups associated to $S$).
- The children of $S \in \mathcal{V}$ are $S \setminus \{a_1\}, \ldots, S \setminus \{a_r\}$, where $a_1, \ldots, a_r$ are the elements of $\mathrm{msg}_{\mathcal{V}}(S)$ which are greater than $\mathrm{F}(S)$.

# The tree of the Arf numerical semigroups as Frobenius variety




$*$ Binary tree: each node has at most two children.

# Frobenius pseudo-varieties (R.-P. and Rosales, 2015)

- The set of numerical semigroups with maximal embedding dimension is not a variety.
- A *Frobenius pseudo-variety* is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfills the following conditions,
  - $\mathcal{P}$ has a maximum element $\Delta(\mathcal{P})$ (with respect to the inclusion order);
  - if $S, T \in \mathcal{P}$, then $S \cap T \in \mathcal{P}$;
  - if $S \in \mathcal{P}$ and $S \neq \Delta(\mathcal{P})$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.
- Examples of pseudo-varieties.
  - The set of numerical semigroups with multiplicity $m$.
  - The set of numerical semigroups with maximal embedding dimension and multiplicity $m$.
  - The set of numerical semigroups admitting a strong admissible pattern and multiplicity $m$: $m$-varieties.
    (Bras-Amorós, García-Sánchez and Vico-Oton, 2013)

# Pseudo-varieties and varieties

- Every variety is a pseudo-variety.
- If $\mathcal{P}$ is a pseudo-variety, then $\mathcal{P}$ is a variety if and only if $\mathbb{N} \in \mathcal{P}$.
- If $\mathcal{P}$ is a family of numerical semigroups with maximum $\Delta$, then $\mathcal{P}$ is a pseudo-variety if and only if $\mathcal{P} \cup \mathcal{C}(\Delta)$ is a variety.

- If $\mathcal{P}$ is a pseudo-variety and $S \in \mathcal{P}$, then $\Delta(\mathcal{P}) \in \mathcal{C}(S)$.
- If $S_1, S_2, \Delta$ are numerical semigroups such that $\Delta \in \mathcal{C}(S_1) \cap \mathcal{C}(S_2)$, then $\Delta \in \mathcal{C}(S_1 \cap S_2)$.

- Let $\mathcal{V}$ be a variety and let $\Delta \in \mathcal{V}$.
  - Then $\mathcal{D}(\mathcal{V}, \Delta) = \{S \in \mathcal{V} \mid \Delta \in \mathcal{C}(S)\}$ is a pseudo-variety.
  - Every pseudo-variety can be obtained in this way.

- $\mathcal{D}(\mathcal{S}, \{0, m, \rightarrow\}) = \{S \in \mathcal{S} \mid S \subseteq \{0, m, \rightarrow\}\}$ is a pseudo-variety.

# $\mathcal{P}$-monoids and $\mathcal{P}$-systems of generators

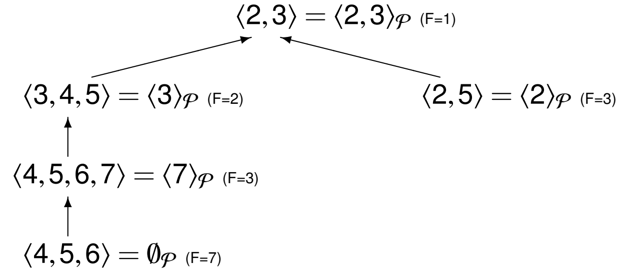
- Let $\mathcal{P}$ be a pseudo-variety.
- A submonoid $M$ of $\mathbb{N}$ is a $\mathcal{P}$-monoid if it can be expressed as intersection of elements in $\mathcal{P}$.
  - The intersection of $\mathcal{V}$-monoids is a $\mathcal{V}$-monoid.
- Let $A \subseteq \Delta(\mathcal{P})$. The *$\mathcal{P}$-monoid generated by $A$* (denoted by $\mathcal{P}(A)$) is the intersection of all the $\mathcal{P}$-monoids containing $A$.
  - $\mathcal{P}(A)$ is the intersection of all elements of $\mathcal{P}$ containing $A$.
- If $M = \mathcal{P}(A)$, then $A$ is a *$\mathcal{P}$-system of generators* of $M$.
- $A$ is a *minimal $\mathcal{P}$-system of generators* of $M$ if $M \neq \mathcal{P}(B)$ for all $B \subsetneq A$.
  - Every $\mathcal{P}$-monoid $M$ has a unique minimal $\mathcal{P}$-system of generators, which in additon is finite. ($A = \mathrm{msg}_{\mathcal{P}}(M)$).
- If $M$ is a $\mathcal{P}$-monoid and $x \in M$, then $M \setminus \{x\}$ is a $\mathcal{P}$-monoid if and only if $x \in \mathrm{msg}_{\mathcal{P}}(M)$.

# The tree of a pseudo-variety

- Let $\mathcal{P}$ be a pseudo-variety (with maximum $\Delta(\mathcal{P})$).
- Let $\mathrm{G}(\mathcal{P})$ be the tree associated to $\mathcal{P}$. We have that
  - the vertices are the elements of $\mathcal{P}$,
  - $(T,S)$ is an edge if $S = T \cup \{\mathrm{F}(T)\}$,
  - $\Delta(\mathcal{P})$ is the root.
- If $S \in \mathcal{P}$, then the unique path connecting $S$ with $\Delta(\mathcal{P})$ is $\mathcal{C}_{\mathcal{P}}(S)$:
  - $\mathrm{C}_{\mathcal{P}}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$, where $S_0 = S$, $S_{i+1} = S_i \cup \{\mathrm{F}(S_i)\}$, for all $i < n$, and $S_n = \Delta(\mathcal{P})$.
- The children of $S \in \mathcal{P}$ are $S \setminus \{a_1\}, \ldots, S \setminus \{a_r\}$, where $a_1, \ldots, a_r$ are the elements of $\mathrm{msg}_{\mathcal{P}}(S)$ which are greater than $\mathrm{F}(S)$.

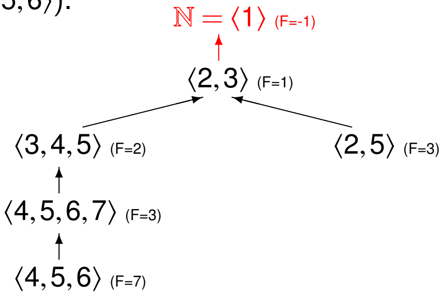
# Example

- Let $T$ be a numerical semigroup, $T \neq \mathbb{N}$.
  - $\mathcal{S}(T,\langle 2,3\rangle) = \{S \in \mathcal{S} \mid T \subseteq S \subseteq \langle 2,3\rangle\}$ is a pseudo-variety with $\Delta(\mathcal{S}(T,\langle 2,3\rangle)) = \langle 2,3\rangle$.
  - If $S \in \mathcal{S}(T,\langle 2,3\rangle)$, then $\mathrm{msg}_{\mathcal{S}(T,\langle 2,3\rangle)}(S) = \mathrm{msg}(S) \setminus T$.
- $\mathrm{G}(\mathcal{P}) = \mathrm{G}(\mathcal{S}(\langle 4,5,6\rangle,\langle 2,3\rangle))$



# Pseudo-varieties trees vs varieties trees

- Let $\mathcal{V}$ be a variety and $\Delta \in \mathcal{V}$.
  - $\mathcal{D}(\mathcal{V},\Delta) = \{S \in \mathcal{V} \mid \Delta \in \mathcal{C}(S)\}$ is a pseudo-variety.
  - $\mathcal{D}(\mathcal{V},\Delta)$ is formed by the descendants of $\Delta$ in the tree $\mathrm{G}(\mathcal{V})$.
  - So, a pseudo-variety is a subtree obtained from the tree of a variety when we take a vertex and all its descendants.
- If $T$ is a numerical semigroup, then $\mathcal{O}(T) = \{S \in \mathcal{S} \mid T \subset S\}$ is a variety.
- The tree of $\mathcal{O}(\langle 4,5,6 \rangle)$.



# Examples: no pseudo-varieties

- Let $\mathcal{R}(\langle 5,6\rangle) = \{S \in \mathcal{S} \mid S \subseteq \langle 5,6\rangle\}$.
  - $T = \langle 5,6\rangle \setminus \{6\} = \langle 5,11,12,18\rangle \in \mathcal{R}(\langle 5,6\rangle)$.
  - $\mathrm{F}(T) = 19$.
  - $T \cup \{19\} \notin \mathcal{R}(\langle 5,6\rangle)$.
  - So, $\mathcal{R}(\langle 5,6\rangle)$ is not a pseudo-variety (or a variety).
- Let $\mathcal{R}(\langle 3,8\rangle, \langle 3,4\rangle) = \{S \in \mathcal{S} \mid \langle 3,8\rangle \subseteq S \subseteq \langle 3,4\rangle\}$.
  - $T = \langle 3,4\rangle \setminus \{4\} = \langle 3,7,8\rangle \in \mathcal{R}(\langle 3,8\rangle, \langle 3,4\rangle)$.
  - $\mathrm{F}(T) = 5$.
  - $T \cup \{5\} \notin \mathcal{R}(\langle 5,6\rangle, \langle 3,5\rangle)$.
  - So, $\mathcal{R}(\langle 3,8\rangle, \langle 3,4\rangle)$ is not a pseudo-variety (or a variety).

# $R$-varieties (R.-P. and Rosales, preprint available at arXiv)

- If $S$ and $T$ are numerical semigroups such that $S \subsetneq T$, then $S \cup \{\max(T \setminus S)\}$ is another numerical semigroup.
- $\mathrm{F}_T(S) = \max(T \setminus S)$ is the *Frobenius number of S restricted to T*.
- An *R-variety* is a non-empty family $\mathcal{R}$ of numerical semigroups that fulfills the following conditions,
  - $\mathcal{R}$ has a maximum element $\Delta(\mathcal{R})$ (with respect to the inclusion order);
  - if $S, T \in \mathcal{R}$, then $S \cap T \in \mathcal{R}$;
  - if $S \in \mathcal{R}$ and $S \neq \Delta(\mathcal{R})$, then $S \cup \{\mathrm{F}_{\Delta(\mathcal{R})}(S)\} \in \mathcal{R}$.

# $R$-varieties, pseudo-varieties and varieties

- Every pseudo-variety is an $R$-variety.
- If $\mathcal{R}$ is an $R$-variety, then $\mathcal{R}$ is a pseudo-variety if and only if $\mathrm{F}(S) \in \Delta(\mathcal{R})$ for all $S \in \mathcal{R}$ such that $S \neq \Delta(\mathcal{R})$.
- If $\mathcal{R}$ is an $R$-variety, then $\mathcal{R}$ is a variety if and only if $\mathbb{N} \in \mathcal{R}$.
- Let $\mathcal{V}$ be a variety and let $T$ be a numerical semigroup.
  - Then $\mathcal{V}_T = \{S \cap T \mid S \in \mathcal{V}\}$ is an $R$-variety.
  - Every $R$-variety is of this form.
- If $\mathcal{R}$ is an $R$-variety and $U$ is a numerical semigroup, then $\mathcal{R}_U = \{S \cap U \mid S \in \mathcal{R}\}$ is an $R$-variety.
- Let $\mathcal{P}$ be a pseudo-variety and let $T$ be a numerical semigroup.
  - Then $\mathcal{P}_T = \{S \cap T \mid S \in \mathcal{P}\}$ is an $R$-variety.
  - Every $R$-variety is of this form.

# $\mathcal{R}$-monoids and $\mathcal{R}$-systems of generators

- Let $\mathcal{R}$ be an $R$-variety.
- A submonoid $M$ of $\mathbb{N}$ is an $\mathcal{R}$-monoid if it can be expressed as intersection of elements in $\mathcal{R}$.
  - The intersection of $\mathcal{R}$-monoids is an $\mathcal{R}$-monoid.
- Let $A \subseteq \Delta(\mathcal{R})$. The *$\mathcal{R}$-monoid generated by $A$* (denoted by $\mathcal{R}(A)$) is the intersection of all the $\mathcal{R}$-monoids containing $A$.
  - $\mathcal{R}(A)$ is the intersection of all elements of $\mathcal{R}$ containing $A$.
- If $M = \mathcal{R}(A)$, then $A$ is an *$\mathcal{R}$-system of generators* of $M$.
- $A$ is a *minimal $\mathcal{P}$-system of generators* of $M$ if $M \neq \mathcal{R}(B)$ for all $B \subsetneq A$.
  - Every $\mathcal{R}$-monoid $M$ has a unique minimal $\mathcal{R}$-system of generators, which in additon is finite. ($A = \mathrm{msg}_{\mathcal{R}}(M)$).
- If $M$ is a $\mathcal{R}$-monoid and $x \in M$, then $M \setminus \{x\}$ is a $\mathcal{R}$-monoid if and only if $x \in \mathrm{msg}_{\mathcal{R}}(M)$.

# The tree of an $R$-variety

- Let $\mathcal{R}$ be an $R$-variety (with maximum $\Delta(\mathcal{R})$).
- Let $\mathrm{G}(\mathcal{R})$ be the tree associated to $\mathcal{R}$. We have that
  - the vertices are the elements of $\mathcal{R}$,
  - $(S,S') \in \mathcal{R}\times\mathcal{R}$ is an edge if $S' = S \cup \{\mathrm{F}_{\Delta(\mathcal{R})}(S)\}$,
  - $\Delta(\mathcal{R})$ is the root.
- If $S \in \mathcal{R}$, then the unique path connecting $S$ with $\Delta(\mathcal{R})$ is $\mathcal{C}_{\mathcal{R}}(S)$:
  - $\mathrm{C}_{\mathcal{R}}(S) = \{S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n\}$, where $S_0 = S$, $S_{i+1} = S_i \cup \{\mathrm{F}_{\Delta(\mathcal{R})}(S_i)\}$, for all $i < n$, and $S_n = \Delta(\mathcal{R})$.
- The children of $S \in \mathcal{R}$ are $S\setminus\{a_1\},\ldots,S\setminus\{a_r\}$, where $a_1,\ldots,a_r$ are the elements of $\mathrm{msg}_{\mathcal{R}}(S)$ which are greater than $\mathrm{F}(S)$.

# Example

- $R(3,8) := \mathcal{R}(\langle 3,8\rangle, \langle 3,4\rangle)$ is an $R$-variety with $\Delta(R(3,8)) = \langle 3,4\rangle$.
  - If $S \in R(3,8)$, then $\mathrm{msg}_{R(3,8)}(S) = \mathrm{msg}(S) \setminus \langle 3,8\rangle$.
- $\mathrm{G}(R(3,8))$

$$
\begin{array}{ll}
\langle 3,4\rangle = \langle 4\rangle_{R(3,8)} & (\mathrm{F}(\langle 3,4\rangle) = 5,\ \mathrm{F}(\langle 3,4\rangle)_\Delta = -1) \\
\uparrow & \\
\langle 3,7,8\rangle = \langle 7\rangle_{R(3,8)} & (\mathrm{F}(\langle 3,7,8\rangle) = 5,\ \mathrm{F}(\langle 3,7,8\rangle)_\Delta = 4) \\
\uparrow & \\
\langle 3,8,10\rangle = \langle 10\rangle_{R(3,8)} & (\mathrm{F}(\langle 3,8,10\rangle) = \mathrm{F}(\langle 3,8,10\rangle)_{\Delta)} = 7) \\
\uparrow & \\
\langle 3,8,13\rangle = \langle 13\rangle_{R(3,8)} & (\mathrm{F}(\langle 3,8,13\rangle) = \mathrm{F}(\langle 3,8,13\rangle)_\Delta = 10) \\
\uparrow & \\
\langle 3,8\rangle = \emptyset_{R(3,8)} & (\mathrm{F}(\langle 3,8\rangle) = \mathrm{F}(\langle 3,8\rangle)_\Delta = 13)
\end{array}
$$

# $R$-varieties trees vs (pseudo-)varieties trees (I)

- Let $\mathcal{R}$ be an $R$-variety and $T \in \mathcal{R}$.
  - $\mathcal{D}(\mathcal{R}, T) = \{S \in \mathcal{R} \mid S$ is a descendant of $T$ in the tree $\mathrm{G}(\mathcal{R})\}$.
  - $\mathcal{D}(\mathcal{R}, T)$ is an $R$-variety.
- Let $\mathcal{V}$ be a variety and $\Delta \in \mathcal{V}$.
  - $\mathcal{D}(\mathcal{V}, \Delta) = \{S \in \mathcal{V} \mid \Delta \in \mathcal{C}(S)\}$ is a pseudo-variety.
  - Every pseudo-variety is of the form $\mathcal{D}(\mathcal{V}, \Delta)$.
- Consequence: there exist $R$-varieties which are not the set formed by all the descendants of an element belonging to a variety.

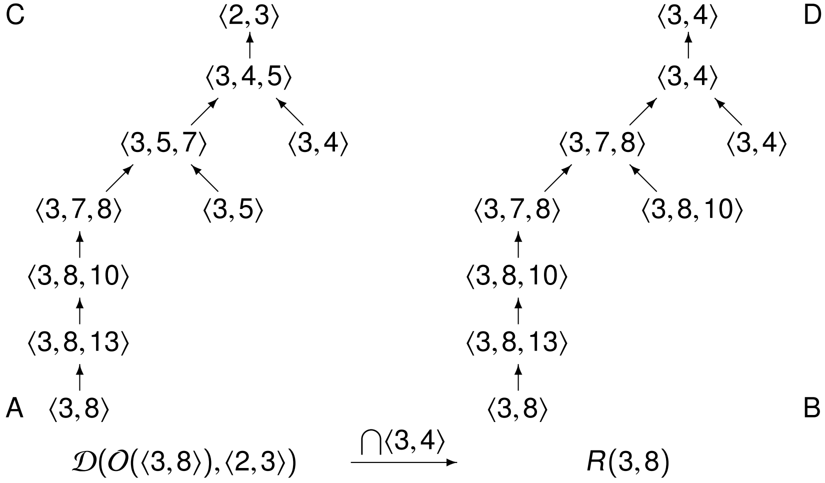
# $R$-varieties trees vs (pseudo-)varieties trees (II)

- Let $\mathcal{V}$ be a variety, let $\Delta \in \mathcal{V}$, and let $T$ be a numerical semigroup.
  - $\mathcal{D}_T(\mathcal{V},\Delta) = \{S \cap T \mid S \in \mathcal{D}(\mathcal{V},\Delta)\}$ is an $R$-variety.
  - Every $R$-variety can be obtained in this way.

- Let $\mathcal{P}$ be a pseudo-variety, let $\Delta \in \mathcal{P}$, and let $T$ be a numerical semigroup.
  - $\mathcal{D}_T(\mathcal{P},\Delta) = \{S \cap T \mid S \in \mathcal{D}(\mathcal{P},\Delta)\}$ is an $R$-variety.
  - Every $R$-variety can be obtained in this way.

# Example

- Let $\mathcal{V} = \mathcal{O}(\langle 3,8\rangle) = \{S \in \mathcal{S} \mid \langle 3,8\rangle \subset S\}$, $\Delta = \langle 2,3\rangle$, and $T = \langle 3,4\rangle$.



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**Thank you very much for your attention!!**