On Critical Binomial Ideals

Ignacio Ojeda Martínez de Castilla
Universidad de Extremadura

(joint work-in-progress with D. Llena and P.A. García Sánchez)

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Semigroup algebra

Let $\mathbb{N}\mathcal{A}$ be a combinatorially finite, cancellative and commutative semigroup with zero element finitely generated by

$$\mathcal{A} := \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

and such that $\mathbb{N}\mathcal{A} \cap (-\mathbb{N}\mathcal{A}) = \{0\}$. For simplicity, we assume that $\mathcal{A} \subseteq \mathbb{N}^d$, for some $d > 0$.

Let $\Bbbk$ be a field and let $\Bbbk[\mathbb{N}\mathcal{A}]$ be the semigroup $\Bbbk$-algebra of $\mathbb{N}\mathcal{A}$, that is to say,

$$\Bbbk[\mathbb{N}\mathcal{A}] = \bigoplus_{\mathbf{a} \in \mathbb{N}\mathcal{A}} \Bbbk\{\mathbf{t}^{\mathbf{a}}\},$$

with the product given by $\mathbf{t}^{\mathbf{a}} \cdot \mathbf{t}^{\mathbf{a}'} = \mathbf{t}^{\mathbf{a}+\mathbf{a}'}$.

Consider the polynomial ring $\Bbbk[\mathbf{X}] := \Bbbk[X_1, \ldots, X_n]$ over $\Bbbk$ generated by the variables $X_1, \ldots, X_n$ in degrees $\mathbf{a}_1, \ldots, \mathbf{a}_n$, respectively.

Semigroup ideal

The kernel of the ring homomorphism

$$\mathbb{k}[\mathbf{X}] = \mathbb{k}[X_1, \ldots, X_n] \xrightarrow{\varphi_0} \mathbb{k}[\mathbb{N}\mathcal{A}];\ X_i \longmapsto \mathbf{t}^{\mathbf{a}_i},$$

is a $\mathbb{N}\mathcal{A}-$homogeneous binomial ideal called the semigroup ideal associated to $\mathcal{A}$. It is clear that $\mathbb{k}[\mathbb{N}\mathcal{A}] = \mathbb{k}[\mathbf{X}]/I_\mathcal{A}$ is $\mathbb{N}\mathcal{A}$-graded.

Any set of generators of $I_\mathcal{A}$ gives a presentation of the semigroup algebra $\mathbb{k}[\mathbb{N}\mathcal{A}]$ as $\mathbb{k}[\mathbf{X}]$-module; more precisely, if $I_\mathcal{A} = \langle f_1, \ldots, f_{\beta_1} \rangle$, then

$$\mathbb{k}[\mathbf{X}]^{\beta_1} \xrightarrow{\varphi_1 = (f_1\ f_2\ \ldots\ f_{\beta_1})} \mathbb{k}[\mathbf{X}] \xrightarrow{\varphi_0} \mathbb{k}[\mathbb{N}\mathcal{A}] \longrightarrow 0$$

is a $\mathbb{N}\mathcal{A}$-graded exact sequence.

Every minimal system of binomial generators of $I_\mathcal{A}$ is $\mathbb{N}\mathcal{A}-$homogeneous and it has the same cardinality.

Critical binomials

A binomial $X_i^{c_i} - \prod_{j \neq i} X_j^{u_{ij}} \in I_\mathcal{A}$ is called critical with respect to X_i if c_i is the least positive integer such that

$$c_i \mathbf{a}_i \in \sum_{j \neq i} \mathbb{N} \mathbf{a}_j.$$

A subideal of $I_\mathcal{A}$ generated by n critical binomials, one for each variable, is said to be a critical binomial ideal associated to $\mathcal{A}$.

Proposition

If $\mathbb{N}\mathcal{A}$ is a numerical semigroup, there exist critical binomial ideals associated to $\mathcal{A}$.

Critical binomials

Proposition

If $\mathbb{Z}\mathcal{A}$ has rank 1, there exist critical binomial ideals associated to $\mathcal{A}$.

However, the converse is not necessarily true.

Example

Let $\mathcal{A} \subset \mathbb{N}^2$ be the set of columns of

$$A = \begin{pmatrix} 3 & 5 & 7 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}.$$

The ideals $I_{\mathcal{A}}$ and $I_{\mathcal{A}'}$ where $\mathcal{A}' = \{15, 25, 35, 22, 33\}$ contain the same critical binomial ideal, $I_{\mathcal{A}}$.

$$I_{\mathcal{A}} = \langle X_2^2 - X_1X_3, X_1^3X_2 - X_3^2, X_4^3 - X_5^2, X_1^4 - X_2X_3\rangle$$

$$I_{\mathcal{A}'} = \langle X_2^2 - X_1X_3, X_1^3X_2 - X_3^2, X_4^3 - X_5^2, X_1^4 - X_2X_3, X_1^2X_2 - X_4X_5\rangle$$

Critical monoids

Definition

We will say that $\mathbb{N}\mathcal{A}$ is a critical monoid if $I_\mathcal{A}$ is a critical binomial ideal.

That is to say, $\mathbb{N}\mathcal{A}$ is a critical monoid if there exists a minimal system of generators of $I_\mathcal{A}$ consisting in critical binomials. In particular, the cardinality of (any) system of binomial generators of $I_\mathcal{A}$ is lesser than or equal to n.

Two problems

- To give necessary and/or sufficient conditions for a semigroup ideal (monoid, resp.) to be a critical binomial ideal (critical monoid, resp.).
- To exhibit families of critial monoids.

Theorem (Herzog, 1970)

If $n = 3$, every numerical semigroup is a critical monoid.

Theorem (Alcántar and Villarreal, 1994)

If $n = 4$ and $I_{\mathcal{A}}$ is critical of height 3, then (after permuting the variables appropriately) $I_{\mathcal{A}}$ has a set of binomial generators of one of the following three types:

(N): $f_1 = X_1^{c_1} - X_3^{u_{13}} X_4^{u_{14}}, f_2 = X_2^{c_2} - X_1^{u_{21}} X_4^{x_{24}}, f_3 = X_3^{c_3} - X_2^{u_{32}} X_4^{u_{34}}$ and $D = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$, with $u_{ij} > 0$, for all i, j.

(gN1): $f_1 = X_1^{c_1} - X_3^{c_3}, f_2 = X_2^{c_2} - X_1^{u_{21}} X_3^{u_{23}}$ and $g = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$.

(gN2): $f_1 = X_1^{c_1} - X_2^{u_{12}} X_3^{u_{13}}, f_2 = X_2^{c_2} - X_1^{u_{21}} X_3^{u_{23}}, D = X_3^{c_3} - X_1^{u_{31}} X_2^{u_{32}}$ and $g = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$, with $u_{ij} > 0$ for all $i \neq 4$ and $u_{4j} \geq 0$, not all zero, for all j.

(N) $f_1 = X_1^{c_1} - X_3^{u_{13}} X_4^{u_{14}}, f_2 = X_2^{c_2} - X_1^{u_{21}} X_4^{X_{24}}, f_3 = X_3^{c_3} - X_2^{u_{32}} X_4^{u_{34}}$ and $D = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$, with $u_{ij} > 0$, for all i, j.

Proposition (Bresinsky, 1988)

Let B be the matrix whose rows are the exponent vectors of f_1, f_2, f_3 and D. i.e.,

$$B = \begin{pmatrix} c_1 & 0 & -u_{13} & -u_{14} \\ -u_{21} & c_2 & 0 & -u_{24} \\ 0 & -u_{32} & c_3 & -u_{34} \\ -u_{41} & -u_{42} & -u_{43} & c_4 \end{pmatrix}.$$

There exists $\mathcal{A} \subset \mathbb{N}$ such that $\mathbb{N}\mathcal{A}$ is a numerical semigroup and $I_{\mathcal{A}} = \langle f_1, f_2, f_3, D \rangle$ if and only if

(a) *the sum of the columns of B are zero;*

(b) *the $(i, j)-$minors of B, $j = 1, \ldots, 4$ are relatively prime for some i.*

Critical ideals of Northcott type

Definition

Let

$$\Phi = \begin{pmatrix} X_1^{u_{n1}} & 0 & \dots & -X_n^{u_{1n}} \\ -X_n^{u_{2n}} & X_2^{u_{n2}} & \dots & 0 \\ 0 & -X_n^{u_{3n}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & X_{n-1}^{u_{nn-1}} \end{pmatrix}$$

and $\mathbf{m} = (X_1^{u_{21}}, \dots, X_{n-2}^{u_{n-1n-2}}, X_{n-1}^{u_{1n-1}})$, with $u_{ij} > 0$, for all i, j. If $(f_1, \dots, f_{n-1})^{\mathrm{T}} = \Phi\,\mathbf{m}^{\mathrm{T}}$ and $D = \det(\Phi)$, then the ideal of $\Bbbk[\mathbf{X}]$ generated by

$$\{f_1, \dots, f_{n-1}, D\}$$

will be called a critical ideal of Northcott type.

Critical ideals of Northcott type

The family of critical ideals of Northcott type contains as subcases the semigroup ideals associated to non-symmetric numerical semigroups of embedding dimension 3 and the family (N) introduced by Alcantar and Villarreal.

The first case has been studied from this point of view by O'Carrol+Planas-Vilanova.

Critical ideals of Northcott type

Theorem

If $J \subset \Bbbk[\mathbf{X}]$ is a critical ideal of Northcott type, then it can be computed $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z} \oplus T$, where T is finite abelian group, such that $I_{\mathcal{A}} = J$.
In this case, we will say that $\mathbb{N}\mathcal{A}$ is a Northcott type semigroup

Observe that Norhtcott type semigroups are critical monoids.

One of the key facts of the proof is that the system of generators $\{f_1, \ldots, f_{n-1}, D\}$ determinated by Φ and $\mathbf{m}$ forms a Groebner basis with respect to a particular term order on $\Bbbk[\mathbf{X}]$

Corollary

Every critical ideal of Northcott type has unique minimal system of binomial generators (up to signs), that is to say, $f_1, \ldots, f_{n-1}$ and D are indispensable binomials.

Apéry Set

Let S be a numerical semigroup, and let z be an integer. The Apéry set of z in S is the set

$$\mathrm{Ap}(S,z) = \{s \in S \mid s - z \notin S\}.$$

Lemma

Let J be a critical ideal of Northcott type and let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be such that $J = I_{\mathcal{A}}$. If $S = \mathbb{N}\mathcal{A}$ is a numerical semigroup, then $\mathrm{Ap}(S, \mathbf{a}_n)$ can be explicitly described.

Therefore, by using the Selmer's formulas:

1. $\mathrm{F}(S) = \max(\mathrm{Ap}(S,s)) - s$,
2. $\mathrm{g}(S) = \frac{1}{s}\sum_{w \in \mathrm{Ap}(S,s)} w - \frac{s-1}{2}$,

the Frobenius number and the genus of S can be explictly determined.

Numerical semigroup

Example

Let J be a critical ideal of Northcott type for $\mathbf{m} = (X_1, \ldots, X_{n-1})$ and let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be such that $J = I_{\mathcal{A}}$. If $\mathbb{N}\mathcal{A}$ is a numerical semigroup, then

- $\mathbf{a}_n = \prod_{i=1}^{n-1}(u_{ni} + 1) - 1$,
- $\mathbf{a}_k = u_{\sigma(k)n} + \big(\sum_{i=1}^{n-2} \prod_{j=1}^{i}(u_{n\,\sigma^k(j)} + 1)\big) u_{\sigma^k(i+1)\,n}$, for $k \neq n$,

and

- $\mathrm{F}(S) = (\sum_{i=1}^{n-1} u_{in} - 1)\mathbf{a}_n - \min \mathbf{a}_j$,
- $\mathrm{g}(S) = (\sum_{i=1}^{n-1} u_{in} - 1)(\mathbf{a}_n - 1)/2$,

Gluings and Nothcott-type ideals

(gN1): $f_1 = X_1^{c_1} - X_3^{c_3}, f_2 = X_2^{c_2} - X_1^{u_{21}} X_3^{u_{23}}$ and $g = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$.

(gN2): $f_1 = X_1^{c_1} - X_2^{u_{12}} X_3^{u_{13}}, f_2 = X_2^{c_2} - X_1^{u_{21}} X_3^{u_{23}}, D = X_3^{c_3} - X_1^{u_{31}} X_2^{u_{32}}$ and $g = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$, with $u_{ij} > 0$ for all $i \neq 4$ and $u_{4j} \geq 0$, not all zero, for all j.

Gluings and Nothcott-type ideals

(gN*): $f_1 = X_1^{c_1} - X_2^{u_{12}} X_3^{u_{13}}$, $f_2 = X_2^{c_2} - X_1^{u_{21}} X_3^{u_{23}}$, $D = X_3^{c_3} - X_1^{u_{31}} X_2^{u_{32}}$ and $g = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$, with ($u_{ij} > 0$ for all $i \neq 4$ and $u_{4j} \geq 0$, not all zero, for all j), or ($u_{12} = u_{32} = 0,\ u_{13} = c_3$ and $u_{31} = c_1$).

Proposition (Bresinsky, 1988)

Let B be the matrix whose rows are the exponent vectors of f_1, f_2 D, and g, that is,

$$B = \begin{pmatrix} c_1 & -u_{12} & -u_{13} & 0 \\ -u_{21} & c_2 & -u_{23} & 0 \\ -u_{31} & -u_{32} & c_3 & 0 \\ -u_{41} & -u_{42} & -u_{43} & c_4 \end{pmatrix}.$$

There exists $\mathcal{A} \subset \mathbb{N}$ such that $\mathbb{N}\mathcal{A}$ is a numerical semigroup and $I_{\mathcal{A}} = \langle f_1, f_2, D, g \rangle$ if and only if

(a) *$\{f_1, f_2, D\}$ generates a toric ideal in $\Bbbk[X_1, X_2, X_3]$.*

(b) *the $(3, j)-$minors of B, $j = 1, \ldots, 4$ are relatively prime.*

Let G be a finitely generated commutative group and let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset G$ be such that $\mathbb{N}\mathcal{A} \cap (-\mathbb{N}\mathcal{A}) = \{0\}$.

Assume that

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

with $\mathcal{A}_1 \neq \varnothing \neq \mathcal{A}_2$.

After reindexing the elements in $\mathcal{A}$ if necessary, we may suppose that $\mathcal{A}_1 = \{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ and $\mathcal{A}_2 = \{\mathbf{a}_{k+1}, \ldots, \mathbf{a}_n\}$.

Definition

We say that $\mathbb{N}\mathcal{A}$ is a gluing of $\mathbb{N}\mathcal{A}_1$ and $\mathbb{N}\mathcal{A}_2$ if $I_{\mathcal{A}}$ has a system of generators of the form $B_1 \cup B_2 \cup \{\mathbf{X}^{\mathbf{u}} - \mathbf{X}^{\mathbf{v}}\}$, where $B_1 \cup \{\mathbf{X}^{\mathbf{u}}\} \subseteq \Bbbk[X_1, \ldots, X_k]$ and $B_2 \cup \{\mathbf{X}^{\mathbf{v}}\} \subseteq \Bbbk[X_{k+1}, \ldots, X_n]$.

Gluings and Nothcott-type ideals

(gN*): $f_1 = X_1^{c_1} - X_2^{u_{12}} X_3^{u_{13}}, f_2 = X_2^{c_2} - X_1^{u_{21}} X_3^{u_{23}}, D = X_3^{c_3} - X_1^{u_{31}} X_2^{u_{32}}$ and $g = X_4^{c_4} - X_1^{u_{41}} X_2^{u_{42}} X_3^{u_{43}}$, with ($u_{ij} > 0$ for all $i \neq 4$ and $u_{4j} \geq 0$, not all zero, for all j), or ($u_{12} = u_{32} = 0,\ u_{13} = c_3$ and $u_{31} = c_1$).

Proposition

There exists $\mathcal{A} = \{a_1, a_2, a_3, a_4\} \subset \mathbb{N}$ such that $\langle f_1, f_2, D, g \rangle = I_{\mathcal{A}}$ if and only if $\mathbb{N}\mathcal{A}$ is a gluing of $c_4 \mathbb{N}\mathcal{A}'$ and $a_4\mathbb{N}$, with $\mathcal{A}' = \{a_i / \gcd(a_1, a_2, a_3) \mid i = 1, 2, 3\}$.

Theorem

Every critical monoid of embedding dimnesion 4 is either a Northcott type semigroup or a gluing of a Northcott type semigroup and $\mathbb{N}$.

Gluings and Nothcott-type ideals

In general, we have the following:

Proposition

Let G be a commutative group, $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset G$ and $\mathbf{a}_{n+1} \in G$. If $\mathbb{N}\mathcal{A}$ is a critical monoid and $S = \mathbb{N}\mathcal{A} + \mathbb{N}\mathbf{a}_{n+1}$ is a gluing of $\mathbb{N}\mathcal{A}$ and $\mathbb{N}\mathbf{a}_{n+1}$, then S is a critical monoid.

Example

A quick search using shows that $S = \langle 11, 13, 14, 15, 19 \rangle$ is not a gluing and not Northcott-type.

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gap> MinimalPresentationOfNumericalSemigroup(s);
    [ [ [ 0, 0, 0, 2, 0 ], [ 1, 0, 0, 0, 1 ] ],
      [ [ 0, 0, 2, 0, 0 ], [ 0, 1, 0, 1, 0 ] ],
      [ [ 0, 2, 0, 0, 0 ], [ 1, 0, 0, 1, 0 ] ],
      [ [ 1, 1, 1, 0, 0 ], [ 0, 0, 0, 0, 2 ] ],
      [ [ 3, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 1 ] ] ]
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Thanks for your attention!