linear inequalities for the hilbert depth of graded modules over polynomial rings

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Introduction

Introduction

- Invariants of Hilbert series $\longrightarrow$ numerical semigroups
- New interpretation of some characterization already explained in Vila-Real and Cortona
- This talk is based on a series of common works with
    - * Lukas Katthän, Goethe-Universität Frankfurt am Main
    - * Jan Uliczka, Universität Osnabrück

  All available on the arXiv.

The setting

Let K be a field.

Let $R := K[X_1, \ldots, X_n]$ be a polynomial ring endowed with a grading, typically

- $\diamond$ standard-$\mathbb{Z}$-grading, i.e., $\deg X_i = 1$
- $\diamond$ nonstandard-$\mathbb{Z}$-grading
- $\diamond$ ($\mathbb{Z}^r$-grading)

Let $0 \neq M = \bigoplus_{\underline{\ell}} M_{\underline{\ell}}$ be a finitely generated graded R-module, with Hilbert series

$$H_M(t) = \sum_{\underline{\ell} \in \mathbb{Z}^r} (\dim_K M_{\underline{\ell}}) \underline{t}^{\underline{\ell}} \in \mathbb{Z}[\![\underline{t}]\!][\underline{t}^{-1}]$$

Series without negative coefficients: nonnegative series.

Previous results

hilbert depth

For the moment, let us restrict ourselves to $\mathbb{Z}$-gradings

Set $d_i := \deg X_i \in \mathbb{N}$ for all $i = 1, \dots n$.

Definition [Hilbert depth]

$$\mathrm{Hdep}(M) := \max\{\mathrm{depth} N \mid N \text{ a f.g. gr. module with } H_N = H_M\}.$$

This is a well-defined but opaque quantity!

Characterizations?

Theorem [—, Uliczka 13]

A formal Laurent series H with denominator $\prod_i (1 - t^{d_i})$ is the Hilbert series of a f.g. graded R-module M if and only if

$$H(t) = \sum_{I \subseteq \{1,\dots,n\}} \frac{Q_I(t)}{\prod_{j \in I}(1 - t^{d_j})}$$

with nonnegative $Q_I(t)$.

Definition [Decomposition Hilbert depth]

$$\operatorname{decHdep}(M) := \max \left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{c} H_M \text{ admits a decompos. as above} \\ \text{with } Q_I = 0 \;\forall\, I \text{ such th. } |I| < r \end{array} \right\}.$$

Case of two variables

Let $R = K[X, Y]$ be with $\alpha := \deg X$, $\beta := \deg Y$ coprime.

Set $\Gamma := \langle \alpha, \beta \rangle$ the numerical semigroup generated by α and β.

Theorem [—, Uliczka 13]

Let M be a finitely generated graded R–module. Then

$\mathrm{Hdep}(M) > 0$ if and only if $H_M(t) = \sum_n h_n t^n$ satisfies the condition

$$\sum_{i \in I} h_{i+n} \leq \sum_{j \in J} h_{j+n}$$

for all $n \in \mathbb{Z}$ and all "fundamental couples" $[I, J]$.

(I) What is a "fundamental couple" $[I, J]$?

Let L be the set of gaps of $\langle\alpha, \beta\rangle$.

An *(α, β)–fundamental couple* $[I, J]$ consists of two integer sequences $I = (i_k)_{k=0}^m$ and $J = (j_k)_{k=0}^m$, such that

(0) $i_0 = 0$.

(1) $i_1, \ldots, i_m, j_1, \ldots, j_{m-1} \in L$ and $j_0, j_m \leq \alpha\beta$.

(2)

$$\begin{array}{llll} i_k \equiv j_k & \bmod \alpha & \text{and} \quad i_k < j_k & \text{for} \quad k = 0, \ldots, m; \\ j_k \equiv i_{k+1} & \bmod \beta & \text{and} \quad j_k > i_{k+1} & \text{for} \quad k = 0, \ldots, m-1; \\ j_m \equiv i_0 & \bmod \beta & \text{and} \quad j_m \geq i_0. & \end{array}$$

(3) $|i_k - i_\ell| \in L$ for $1 \leq k < \ell \leq m$.

(II) What is a "fundamental couple" $[I, J]$?

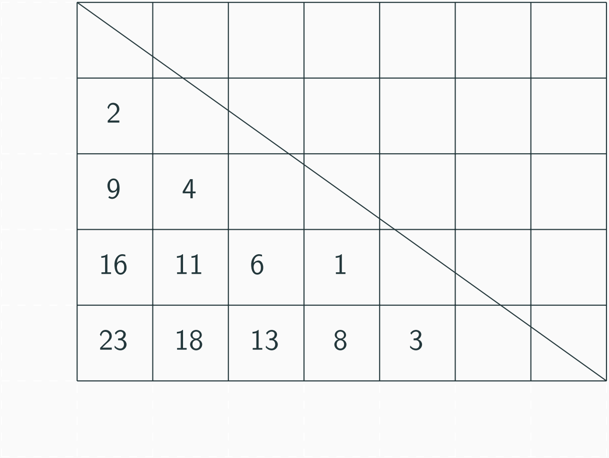
- I consists of minimal generators of "relative ideals" $=$ "semimodules" Δ of Γ.
- J contains "small shifts" of I-sets which turn out to generate a sort of syzygy Syz_{Δ}.

Syzygy in the sense that any element in Syz_{Δ} admits more than one presentation in the form $i + x$ with $i \in I$ and $x \in \Gamma$.

In the special case $\Gamma = \langle 3, 5 \rangle$ the criterion is given by the inequalities

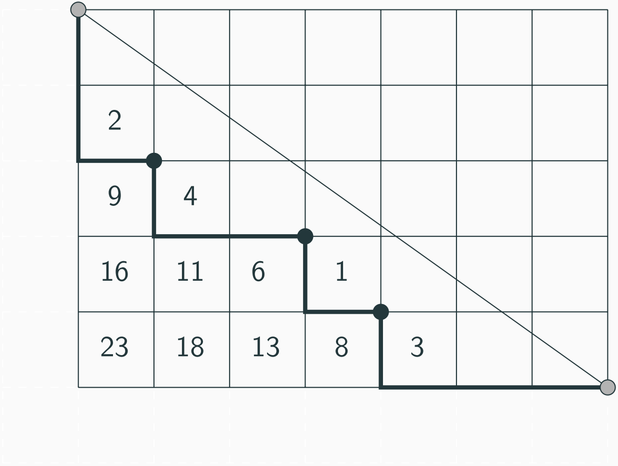
$$\begin{aligned}
h_{n+0} &\leq h_{n+15}, \\
h_{n+0} + h_{n+1} &\leq h_{n+6} + h_{n+10}, \\
h_{n+0} + h_{n+2} &\leq h_{n+12} + h_{n+5}, \\
h_{n+0} + h_{n+4} &\leq h_{n+9} + h_{n+10}, \\
h_{n+0} + h_{n+7} &\leq h_{n+12} + h_{n+10}, \\
h_{n+0} + h_{n+1} + h_{n+2} &\leq h_{n+5} + h_{n+6} + h_{n+7}, \\
h_{n+0} + h_{n+2} + h_{n+4} &\leq h_{n+5} + h_{n+7} + h_{n+9}
\end{aligned}$$

Lattice paths for $\Gamma = \langle 5, 7 \rangle$



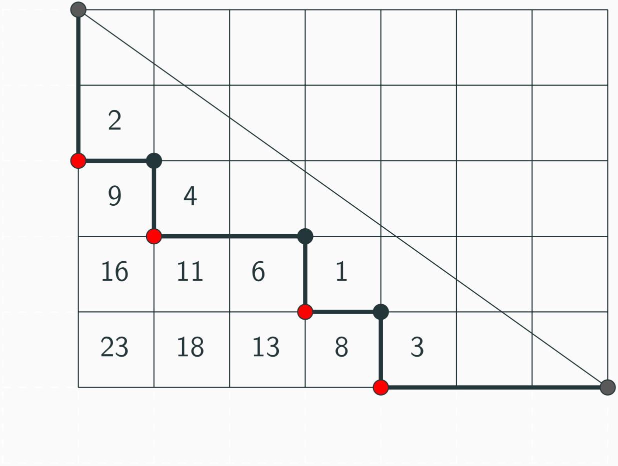
$J = [15, 13, 16, 14]$.

Lattice paths for $\Gamma = \langle 5, 7 \rangle$



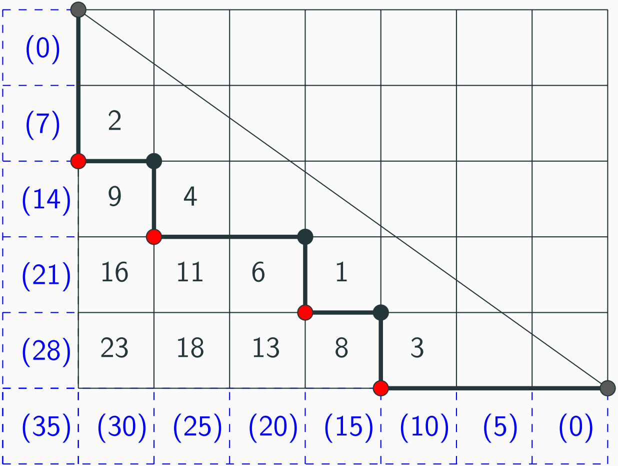
$I = [0, 8, 6, 9]$.

Lattice paths for $\Gamma = \langle 5, 7 \rangle$



$$I = [0, 8, 6, 9].$$

Lattice paths for $\Gamma = \langle 5, 7 \rangle$



$I = [0, 8, 6, 9]$ and $J = [15, 13, 16, 14]$.

New results

A deep algebraic meaning of the inequalities
$\sum_{i\in I} h_{i+n} \leq \sum_{j\in J} h_{j+n}$ remained rather hidden.

New insights appeared when considering the $\mathbb{Z}^r$-grading.

The starting question arose by looking at the decomposition theorem (already mentioned):

A formal Laurent series H with denominator $\prod_i(1-t^{d_i})$ is the Hilbert series of a f.g. graded R-module M iff

$$H(t) = \sum_{I\subseteq\{1,\dots,n\}} \frac{Q_I(t)}{\prod_{j\in I}(1-t^{d_j})} \quad \text{with nonnegative } Q_I.$$

Question: Is the condition of the Thm satisfied by *every* rational function with the given denominator and nonnegative coefficients?

[excursus]

Question: Which formal Laurent series arise as Hilbert series of R-modules (in a certain class)?

Conditions: The series must...

- ... have nonnegative coefficients.
- ... be rational function with denominator $\prod_i (1 - \underline{t}^{\deg X_i})$.
- ...

Related work:

- Macaulay, 1927: cyclic modules, standard $\mathbb{Z}$-grading.
- Boij & Smith, 2015: modules generated in degree 0, standard $\mathbb{Z}$-grading + technical details

Theorem [Katthän, —, Uliczka 2016]

Let $H \in \mathbb{Z}[\![t]\!][t^{-1}]$ be a formal Laurent series, which is the Hilbert series of some finitely generated graded R-module M. Let further $S := R/(X^\beta - Y^\alpha)$.

Then the following statements are equivalent:

(a) $\mathrm{Hdep}(M) > 0$

(b) For any finitely generated torsionfree S-module N, it holds that

$$\frac{H \cdot H_N}{H_R} \geq 0.$$

(c) Condition (b) holds for any finitely generated torsionfree S-module of rank 1.

(d) For all $n \in \mathbb{Z}$, $[I, J]$ fundamental couple, $H = \sum_i h_i t^i$ satisfies

$$\sum_{i \in I} h_{i+n} \leq \sum_{j \in J} h_{j+n} \qquad (\star)$$

We need the following result about the structure of fundamental couples.

Lemma

Let $[I = (i_k), J = (j_k)]$ be a fundamental couple of length m. Then there exist two integer sequences

$$\beta > a_0 > a_1 > \cdots > a_m = 0, \quad \text{and}$$
$$0 = b_0 < b_1 < \cdots < b_m < \alpha$$

such that

$$i_k = \alpha\beta - a_{k-1}\alpha - b_k\beta \quad \text{for } 1 \leq k \leq m, \;\text{ and}$$
$$j_k = \alpha\beta - a_k\alpha - b_k\beta \qquad \text{for } 0 \leq k \leq m$$

(c) $\Rightarrow$ (d): Let $[I, J]$ be a fundamental couple.

Recall that $S = K[t^\alpha, t^\beta]$ is the monoid algebra of Γ. Let $N \subseteq K[t]$ be the S-module generated by $t^{\alpha\beta - j_0}, \dots, t^{\alpha\beta - j_m}$.

This module is torsionfree, hence $\frac{H_M H_N}{H_R} \geq 0$ by assumption.

To see that this inequality implies $(\star)$, we need to compute H_N.

Let $(a_k)_{k=0}^m, (b_k)_{k=0}^m$ be the sequences as in Lemma and let

$$\tilde{N} := (X^{a_0} Y^{b_0}, \dots, X^{a_m} Y^{b_m}).$$

It is easy to see that $\tilde{N}$ is the preimage of N under the projection $R \to S$.

In particular, note that $X^\beta - Y^\alpha \in \tilde{N}$, because $X^{a_0}, Y^{b_m} \in \tilde{N}$.

Hence $N \cong \tilde{N}/(X^\beta - Y^\alpha)$ and thus $H_N = H_{\tilde{N}} - t^{\alpha\beta} H_R$.

By considering the minimal free resolution of $\tilde{N}$, one sees that its syzygies are generated in the degrees $a_{k-1}\alpha + b_k\beta$ for $1 \leq k \leq m$.

Therefore

$$\frac{H_N}{H_R} = \frac{H_{\tilde{N}} - t^{\alpha\beta} H_R}{H_R} = \sum_{k=0}^{m} t^{a_k\alpha + b_k\beta} - \sum_{k=1}^{m} t^{a_{k-1}\alpha + b_k\beta} - t^{\alpha\beta}$$

$$= \sum_{k=0}^{m} t^{\alpha\beta - j_k} - \sum_{k=1}^{m} t^{\alpha\beta - i_k} - t^{\alpha\beta - i_0} = t^{\alpha\beta} \left(\sum_{j \in J} t^{-j} - \sum_{i \in I} t^{-i} \right)$$

Then we obtain

$$0 \leq \frac{H \cdot H_N}{H_R} = \left(\sum_{n \in \mathbb{Z}} h_n t^n\right) t^{\alpha\beta} \left(\sum_{j \in J} t^{-j} - \sum_{i \in I} t^{-i} \right)$$

$$= t^{\alpha\beta} \sum_{n \in \mathbb{Z}} t^n \left(\sum_{j \in J} h_{n+j} - \sum_{i \in I} h_{n+i} \right),$$

and $(\star)$ is satisfied for $[I, J]$.