On divisor-closed submonoids and minimal distances in finitely generated monoids

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- If H is an affine semigroup, we give a geometrical characterization of such submonoids in terms of its cone.
- Algorithm for computing $\Delta^*(H)$.

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Definition (Rosales-García, 99)

A semigroup H is an **Archimedean semigroup** if for every $(x, y) \in H \times H$, with $x \neq y$, there exit $k \in \mathbb{N} \setminus \{0\}$ and $z \in H$ such that $kx = y + z$.

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- Let H be a monoid. The Archimedean components of H are subsemigroups of H.
- Let H be a finitely generated monoid. Then $H/\mathcal{N}$ is a finite monoid.
- Every finitely generated monoid is a finite lattice of Arquimedean semigroups.

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- $S_2 = \langle (5, 0) \rangle \subseteq \mathbb{N}^2$ is not divisor-closed submonoid, $(2, 0) + (3, 0) = (5, 0) \in S_2$ but $(2, 0) \notin S_2$.

Proposition

Let H be a finitely generated monoid with $G = \{g_1, \ldots, g_p\}$ *one of its system of generators. Then, every divisor-closed submonoid of H is finitely generated and has a system of generators contained in G.*

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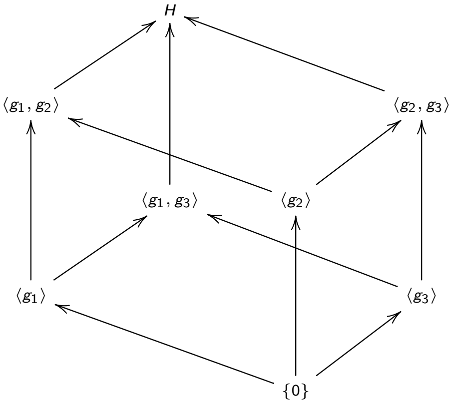

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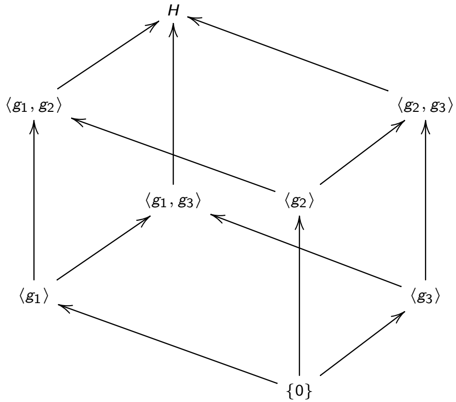

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S is not divisor-closed submonoid.

S **is divisor-closed?**, Arquimedean components.

- They are semigroups. They are not monoids.
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Arquimedean Components of $(\mathbb{N}^2, +)$

- $C_0 = (0,0)$, $C_1 = \{(x_1,x_2) | x_1 \geq 1 \text{ and } x_2 \geq 1\}$,
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$C_1 \cup \{(0,0)\}$, $C_2 \cup \{(0,0)\}$ and $C_3 \cup \{(0,0)\}$, monoids.
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- *Numerical semigroups not have any non-trivial divisor-closed submonoids.*
- *The set of divisor-closed submonoids of a monoid is a finite lattice with respect to inclusion.*

Divisor-closed submonoids of affine semigroups

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Let H be an affine semigroup of $\mathbb{N}^n$. Define the **rational cone** of H as $\mathrm{L}_{\mathbb{Q}_+}(H) = \{\sum_{i=1}^{r} \lambda_i h_i \mid r \in \mathbb{N}, h_i \in H, \lambda_i \in \mathbb{Q}_+\}$.

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$$\inf(\mathfrak{A}) = \cap\{F \in \mathfrak{F}(\mathbf{C}) | F \in \mathfrak{A}\}$$
$$\sup(\mathfrak{A}) = \cap\{G \in \mathfrak{F}(\mathbf{C}) | \forall F \in \mathfrak{A} :\ F \subset G\},$$

for every $\mathfrak{A} \subset \mathfrak{F}(\mathbf{C})$.

Theorem

Let $H \subset \mathbb{N}^n$ be an affine semigroup and let S be a submonoid of H. Then, S is a divisor-closed submonoid of H if and only if there exists a face F of $\mathrm{L}_{\mathbb{Q}_+}(H)$ such that $S = F \cap H$.

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Corollary

Let H be an affine semigroup of $\mathbb{N}^n$. The lattice of divisor-closed submonoids of H, the lattice of Archimedean components of H and the lattice of faces of the cone $\mathrm{L}_{\mathbb{Q}_+}(H)$ are isomorphic.

Definition

An affine semigroup $H \subset \mathbb{N}^n$ is **simplicial** if the cone $\mathrm{L}_{\mathbb{Q}_+}(H)$ is generated by n linearly generators of H.

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Corollary

Let H be a simplicial affine submonoid of $\mathbb{N}^n$. The number of divisor-closed submonoids of H is equal to 2^n.

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- $\pi(H) = \widetilde{H}$ and
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- If H is numerical semigroup $\implies \Delta^*(H) = \{\min(\Delta(H))\}$.

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Lemma

Let $H = \langle h_1, \ldots, h_p \rangle \cong \mathbb{N}^p / \sim_M$ be a monoid with $\{m_1, \ldots, m_r\}$ a system of generators of M. Then

$$\min(\Delta(H)) = \min\{|m| : |m| > 0,\ m \in M\} = \gcd(|m_1|, \ldots, |m_r|).$$

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Algorithm

Input: $H \cong \mathbb{N}^p / \sim_M$. Output: $\Delta^*(H)$.

1. *Compute the lattice $\mathfrak{F}(H)$ of divisor-closed submonoids of H.*
2. *For every $S \in \mathfrak{F}(H)$, if $\{[e_{i_1}]_{\sim_M}, \ldots, [e_{i_t}]_{\sim_M}\}$ is a system of generators of S, compute a system of generators G_S of the group obtained from the intersection of M with $\{(x_1, \ldots, x_p) \in \mathbb{Z}^p \mid x_i = 0 \text{ for all } i \notin \{i_1, \ldots, i_t\}\}$.*
3. *For every $S \in \mathfrak{F}(H)$, compute $|G_S| = \{\sum_{i=1}^{p} |m_i| : (m_1, \ldots, m_p) \in G_S\}$ and $\mathrm{d}_S = \gcd(|G_S|)$.*
4. *Return $\{d_S \mid S \in \mathfrak{F}(H)\}$.*

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Therefore $\Delta^*(H) = \{4, 8\}$.

Thanks for your attention!!