Cyclotomic Numerical Semigroups II –Polynomials playing pingpong–

Pieter Moree,
Max Planck Institute for Mathematics, Bonn

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Overview

1. The pingpong players: $P_S(x)$ and $\Phi_n(x)$
2. First match
   - Semigroup polynomial $P_{\langle p,q\rangle}(x)$
   - Binary cyclotomic polynomials
   - Exponent gaps
   - Gapblocks
3. Second match
   - General cyclotomic polynomials
   - Cyclotomic numerical semigroups
   - Symmetric non-cyclotomic numerical semigroups
   - Counting cyclotomic semigroups of given Frobenius number
4. Polynomially related numerical semigroups
   - An Application

Papers to be discussed

- E.-A. Ciolan, P.A. García-Sánchez and P. Moree, Cyclotomic numerical semigroups, *SIAM J. Discrete Math.* **30** (2016), 650–668
- Cyclotomic numerical semigroups. II, in preparation.
- Pedestrian: P. Moree, Numerical semigroups, cyclotomic polynomials and Bernoulli numbers, *Amer. Math. Monthly* **121** (2014), 890–902.
- O.-M. Camburu, E.-A. Ciolan, F. Luca, P. Moree and I.E. Shparlinski, Cyclotomic coefficients: gaps and jumps, *J. Number Theory*, **163** (2016), 211–237
- H. Hong, E. Lee, H.-S. Lee and C. Park, Maximum gap in (inverse) cyclotomic polynomial, *J. Number Theory* **132** (2012), 2297–2315
- P. Moree, Inverse cyclotomic polynomials, *J. Number Theory* **129** (2009), 667–680
- Some other results from older papers by the speaker (and Y. Gallot)

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Corollary

The nonzero coefficients of $P_S(x)$ alternate between 1 and -1.

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$F(\langle p, q\rangle) = \deg(\Phi_{pq}(X)) - 1 = (p-1)(q-1) - 1 = pq - p - q.$

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Corollary (Migotti, 1887)

Coefficients of $\Phi_{pq}(x)$ *are in* $\{-1, 0, 1\}$.

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Lemma

$$a_{pq}(m) = \begin{cases} 1 & \textit{if } m = ip + jq \textit{ with } 0 \le i \le \rho - 1,\ 0 \le j \le \sigma - 1; \\ -1 & \textit{if } m = ip + jq - pq \textit{ with } \rho \le i \le q - 1,\ \sigma \le j \le p - 1; \\ 0 & \textit{otherwise.} \end{cases}$$

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$\rho = 3^{-1} \pmod 5 = 2$, $\sigma = 5^{-1} \pmod 3 = 2$,
$g(\langle p, q\rangle) = (p-1)(q-1)/2$

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Given $f(x) = c_1x^{e_1} + \cdots + c_tx^{e_t} \in \mathbb{Z}[x]$, with $c_i \neq 0$ and $e_1 < \cdots < e_t$, we define the maximum gap of f as

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- More manageable when turned into a problem involving the maximum gaps of inverse cyclotomic polynomials.

Inverse cyclotomic polynomials

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Theorem (Moree, JNTh, 2009)

We have $B(pqr) \leq p - 1$ and equality holds if and only if

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Conjecturally $A(pqr) \leq 2p/3$.

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Claimed without proof that $\mathcal{R}_3(x) = o(\mathcal{Q}_3(x))$,
where $\mathcal{R}_3(x) = \#\{n \in \mathcal{R}_3 : n \leq x\}$ and $\mathcal{Q}_3(x)$ is defined similarly.

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Camburu, Ciolan, Luca, M., Shparlinski

$$\mathcal{R}_3(x)=\frac{cx}{(\log x)^2}+O\left(\frac{x\log\log x}{(\log x)^3}\right),\ c=(1+\log 4)\log 4.$$

Exponent gaps

$$g(\Phi_p)=1,\quad g(\Psi_p)=1,\quad g(\Phi_{pq})=p-1,\quad g(\Psi_{pq})=q-p+1$$

Hong-Lee-Lee-Park

Put $\mathcal{Q}_3=\{n=pqr:\quad 2<p<q<r \text{ primes}\}$ (ternary integers)

Put $\mathcal{R}_3=\{n\in\mathcal{Q}_3:\quad 4(p-1)>q,\quad p^2>r\}$

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Compare witht the classical estimate (Gauss, Landau)

$$\mathcal{Q}_3(x)=(1+o(1))\frac{x(\log\log x)^2}{2\log x}.$$

Gapblocks

Lemma

Let $p < q$ be primes. Then $g(\Phi_{pq}) = p - 1$.

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Since $S = \langle p, q \rangle$ is symmetric, there is a one to one correspondence between k-gapblocks and k-elementblocks. We have that $g(\Phi_{pq})$ equals the largest gap block in S. Presence of $\langle p \rangle$ in $S = \langle p, q \rangle$ ensures that $g(\Phi_{pq}) \leq p - 1$. Since $S = \{1, p, \ldots\}$, we have $g(\Phi_{pq}) = p - 1$. □

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(i) $g(\Phi_{pq}) = p - 1$ *and the number of maximum gaps equals* $2[q/p]$;

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(i) *$g(\Phi_{pq}) = p - 1$ and the number of maximum gaps equals $2[q/p]$;*

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The notation $\{0\}_m$ indicates a string $\underbrace{0, \ldots, 0}_{m}$ of m consecutive zeros.

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$$P_S(x) = \frac{(1-x)(1-x^{ab})}{(1-x^a)(1-x^b)} = \prod_{d|ab,\ d\nmid a, d\nmid b} \Phi_d(x)$$

is an inclusion-exclusion polynomial (Bachman, 2010).

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Let $2 \leq a < b$ be coprime positive integers. Then

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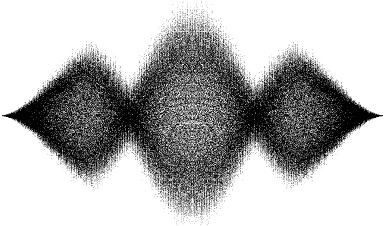
$\Phi_n(x)$ with more than two prime factors

$\Phi_n(x)$ with $n = 4849845 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$



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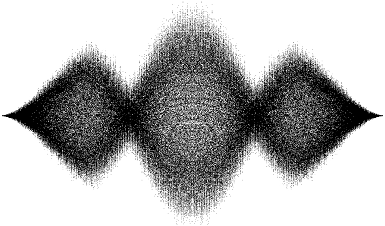

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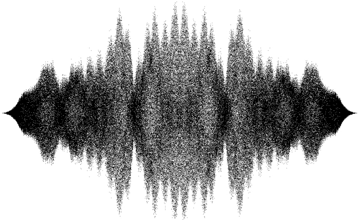

Calculation of $\Phi_n(1)$

Lemma (Value at 1)

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Calculation of $\Phi_n(\zeta)$ with ζ a general root of unity.
Not much known. Work in progress.

Consequences for cyclotomic ns

As we have seen, if a NS is cyclotomic, then

$$P_S(x) = \prod_{d \in \mathcal{D}} \Phi_d(x)^{e_d}, \text{ with } e_d > 0 \text{ uniquely determined.}$$

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Since $P_S(1) = 1$ and $\Phi_1(x) = x - 1$ we infer that $e_1 = 0$.

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Proof.

Since $P_S(1) = 1$ and $\Phi_1(x) = x - 1$ we infer that $e_1 = 0$. Let p^m be a prime power in $\mathcal{D}$. Then by the value at 1 lemma we have $p | \Phi_{p^m}(1) | P_S(1)$. Contradiction. □

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Lemma (Connection with genus)

Let $S \neq \mathbb{N}$ *be a numerical semigroup. Then* $P_S'(1) = g(S)$.

Semigroup Polynomials

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The conclusion now follows on comparing (1) and (2). □

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Lemma

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Example. Take $S = \langle 6, 9, 11 \rangle$. Then $P_S(-1) = 3$ and $P_S = \Phi_{18}\Phi_{33}$.

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- We took all numerical semigroups S that are symmetric and not complete intersection with $F(S) \le k$ and determined how often on average Lemma "Even beats odd" applies. Our computations (with $k \le 69$) indicate that likely an average exists and is in $[0.8, 0.85]$.

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Work in progress...

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Example

$S = \langle 5, 6, 7, 8 \rangle$, with $F(S) = 9$ is the symmetric ns with the smallest Frobenius number that is not cyclotomic.

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Conjecture

Put $P_{S_k}(x) = 1 - x + x^k - x^{2k-1} + x^{2k}$. For every $k \geq 5$ this polynomial has a root not on the unit circle.

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They use results on linear relations between roots of unity.

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It follows that there are abundantly many symmetric numerical semigroups that are not cyclotomic.

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Theorem (Boyd and Montgomery, 1988)

$$c(n) \sim A \frac{e^{B\sqrt{n}}}{n\sqrt{\log n}}, \;\; n \to \infty.$$

Polynomially Related Numerical Semigroups

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We say that the numerical semigroup S is polynomially related to the numerical semigroup T, and denote this by $S \leq_P T$, if there exist $f(x) \in \mathbb{Z}[x]$ and an integer $w \geq 1$ such that

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Example

a) $\langle p^a, q^b \rangle \leq_P \langle p^m, q^n \rangle$ if $1 \leq a \leq m$ and $1 \leq b \leq n$.

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Problem

Find necessary and sufficient conditions such that $S \leq_P T$.

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In proving the following, we make repeated use of the fact that $P_S(1) = 1$ and $P'_S(1) = g(S)$.

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Lemma

Suppose that $H_S(x^w)f(x) = H_T(x)$ holds with S, T numerical semigroups. Then

a) $f(0) = 1$.
b) $f(1) = w$.
c) $f'(1) = w(g(T) - wg(S) + (w-1)/2)$.
d) $F(T) = wF(S) + \deg f$.
e) *If w is even, then $f(-1) = 0$.*
f) *If w is odd, then $f(-1) = P_T(-1)/P_S(-1)$.*
g) *If T is cyclotomic, then so is S.*
h) *If S is cyclotomic, then T is cyclotomic iff f is Kronecker.*

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Let $p \neq q$ be primes and m, n positive integers. The quotient

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In fact, a more general result holds.

Theorem

Suppose that S and T are numerical semigroups with $H_S(x^w)f(x) = H_T(x)$ for some $w \geq 1$ and $f \in \mathbb{N}[x]$. Put $Q(x) = P_T(x)/P_S(x^w)$. Then $Q(0) = 1$ and $Q(x)$ is a monic polynomial having non-zero coefficients that alternate between 1 and -1.

Thank you for attention!