Delta Set for Numerical Semigroup with Embedding Dimension 3

David Llena Carrasco

Departament of Mathematics
University Of Almería

International Meeting on Numerical Semigroup with Applications
Levico (Trento), July 4-8, 2016

This is a joint work with

- Pedro García Sánchez (Universidad de Granada)
- Alessio Moscariello (Università di Catania)

The talk is based in two papers:

- Delta Sets for numerical semigroups with embedding dimension three, arXiv:1504.02116
- Delta Sets for symmetric numerical semigroups with embedding dimension three, in progress

Numerical Semigroups with embedding dimension three

The numerical semigroups we consider here have embedding dimension three.

$$S = \langle n_1, n_2, n_3 \rangle \subset \mathbb{N} \text{ with } gcd(n_1, n_2, n_3) = 1$$

$$S = \{a_1 n_1 + a_2 n_2 + a_3 n_3 \mid a_1, a_2, a_3 \in \mathbb{N} \cup \{0\}\}$$

Factorizations of an element $s \in S$

$Z(s) = \{(z_1, z_2, z_3) \in \mathbb{N}^3 \mid \text{ with } s = z_1 n_1 + z_2 n_2 + z_3 n_3\}$

Length of a factorization $\mathbf{z} = (z_1, z_2, z_3)$

$\ell(\mathbf{z}) = z_1 + z_2 + z_3$

Sets of length of factorizations of $s \in S$

$\mathsf{L}(s) = \{\ell(\mathbf{z}) \mid \mathbf{z} \in Z(s)\}, \ s \in S$

Delta Sets

Delta Set

We order the set $\mathsf{L}(s)$ which is always finite

$$\mathsf{L}(s) = \{l_1 < l_2 < \cdots < l_n\}$$

And define the Delta sets as

- $\Delta(s) = \{l_i - l_{i-1} \mid i = 2, \ldots, n\}$.
- $\Delta(S) = \cup_{s \in S} \Delta(s)$.

We will focus in the set $\Delta(S)$.

Geroldinger (1991)

Let S be a numerical semigroup, then

$$\min \Delta(S) = \gcd \Delta(S).$$

Set $d = \gcd \Delta(S)$. There exists $k \in \mathbb{N} \setminus \{0\}$ such that

$$\Delta(S) \subseteq \{d, 2d, \ldots, kd\}.$$

Example

Let $S = \langle 3, 5, 7 \rangle = \{0, 3, 5, 6, 7, 8, 9, 10, 11, \ldots\}$

In this case, except $0, 3, 5, 6, 7, 8, 9, 11$, the other elements in S have more than one factorization.

$$\begin{array}{rclrcl} \mathsf{Z}(10) & = & \{(1,0,1),(0,2,0)\} & \mathsf{L}(10) & = & \{2\} \\ \mathsf{Z}(12) & = & \{(0,1,1),(4,0,0)\} & \mathsf{L}(12) & = & \{2,4\} \\ \mathsf{Z}(14) & = & \{(0,0,2),(3,1,0)\} & \mathsf{L}(14) & = & \{2,4\} \end{array}$$

$\mathsf{Z}(30) = \{(0,6,0),(1,4,1),(2,2,2),(3,0,3),(5,3,0),(6,1,1),(10,0,0)\}$
$\mathsf{L}(30) = \{6, 8, 10\}$

$$\Delta(10) = \emptyset, \quad \Delta(12) = \{2\}, \quad \Delta(14) = \{2\}, \quad \Delta(30) = \{2\}$$

The aim of this work is to prove that $\Delta(S)$ can be constructed from only two elements, and then we give and fast algorithm to compute it.

The Betti elements and the M_S group

Betti elements

For $s \in S$ we consider a graph

- Vertices are elements $\mathbf{z}$ in $\mathsf{Z}(s)$
- There exists an edge between $\mathbf{z}$ and $\mathbf{z}'$ if and only if $\mathbf{z} \cdot \mathbf{z}' \neq 0$

We say that $s \in S$ is a Betti element if its graph is not connected.

For embedding dimension 3, $\#\mathrm{Betti}(S) \in \{1, 2, 3\}$

In the last example $\mathrm{Betti}(\langle 3, 5, 7\rangle) = \{10, 12, 14\}$.
$\mathsf{Z}(10) = \{(1, 0, 1), (0, 2, 0)\}$, $\mathsf{Z}(12) = \{(4, 0, 0), (0, 1, 1)\}$, $\mathsf{Z}(14) = \{(3, 1, 0), (0, 0, 2)\}$

The group associated to a numerical semigroup

- $M_S = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 n_1 + x_2 n_2 + x_3 n_3 = 0\}$.
- $\mathbf{v}_1 = (4, -1, -1)$ and $\mathbf{v}_2 = (3, 1, -2)$ span M_S as a group.
- $\delta_1 = \ell(\mathbf{v}_1)$ and $\delta_2 = \ell(\mathbf{v}_2)$.

The Euclid's set

For δ_1 and δ_2 non-negative coprime integer, define

$$\eta_1 = \max\{\delta_1, \delta_2\}, \eta_2 = \min\{\delta_1, \delta_2\}, \text{ and } \eta_3 = \eta_1 \bmod \eta_2$$

In general for $i > 2$, $\eta_{i+2} = \eta_i - \left\lfloor \frac{\eta_i}{\eta_{i+1}} \right\rfloor \eta_{i+1} = \eta_i \bmod \eta_{i+1}$. As in Euclid's algorithm.

Euclid's set

Set
$\mathrm{D}(\eta_1, \eta_2) = \{\eta_1, \eta_1 - \eta_2, \ldots, \eta_1 \bmod \eta_2 = \eta_3\}$,
$\mathrm{D}(\eta_2, \eta_3) = \{\eta_2, \eta_2 - \eta_3, \ldots, \eta_2 \bmod \eta_3 = \eta_4\}$,
$\mathrm{D}(\eta_3, \eta_4) = \{\eta_3 - \eta_4, \ldots, \eta_3 \bmod \eta_4 = \eta_5\}$,
$\ldots$
$\mathrm{D}(\eta_i, \eta_{i+1}) = \{\eta_i - \eta_{i+1}, \ldots, \eta_i \bmod \eta_{i+1} = \eta_{i+2} = 0\}$.

The Euclid's set for δ_1 and δ_2 is

$$\mathrm{Euc}(\delta_1, \delta_2) = \bigcup_{i \in I} \mathrm{D}(\eta_i, \eta_{i+1})$$

Theorem

For $S = \langle n_1, n_2, n_3 \rangle$ we have:

$$\bigcup_{s\in S} \Delta(s) = \Delta(S) = \mathrm{Euc}(\delta_1, \delta_2) = \bigcup_{i\in I} \mathrm{D}(\eta_i, \eta_{i+1})$$

Moreover, for every $\delta_1 \neq \delta_2$ there exists a numerical semigroup with $\Delta(S) = \mathrm{Euc}(\delta_1, \delta_2)$.

This result does not hold true for higher embedding dimensions.

Corollary

As a consequence of the above result, if $1 \in \Delta(S)$, then $\{2, 3\} \in \Delta(S)$.

This solves a conjecture proposed by Chapman in the three generated case.

More about the Betti set for $S = \langle n_1, n_2, n_3 \rangle$

We know that, in our setting, M_S is spanned by two vectors, say $\mathbf{v}_1, \mathbf{v}_2$.

We going to define $\mathbf{v}_1, \mathbf{v}_2 \in M_S$ depending on $\#\text{Betti}(S)$.

$\#\text{Betti}(S)$	1	2
$\langle n_1, n_2, n_3 \rangle$	$\langle s_2 s_3, s_1 s_3, s_1 s_2 \rangle$	$\langle am_1, am_2, bm_1 + cm_2 \rangle$
$\text{Betti}(S)$	$\{s_1 s_2 s_3\}$	$\{am_1 m_2, a(bm_1 + cm_2)\}$
Z($betti_1$)	$\{(s_1, 0, 0), (0, s_2, 0), (0, 0, s_3)\}$	$\{(m_2, 0, 0), (0, m_1, 0)\}$
	$s_1 > s_2 > s_3$	$m_2 > m_1$
Z($betti_2$)		$\{(b, c, 0), (b + m_2, c - m_1, 0), \ldots$ $(b + im_2, c - im_1, 0), (b - m_2, c + m_1, 0) \ldots$ $(b - jm_2, c + jm_1, 0), (0, 0, a)\}$
Z($betti_3$)		
$\mathbf{v}_1$	$(s_1, -s_2, 0) = (+, -, 0)$	$(m_2, -m_1, 0) = (+, -, 0)$
$\mathbf{v}_2$	$(0, s_2, -s_3) = (0, +, -)$	$(b + \lambda m_2, c - \lambda m_1, -a) = (+, +, -)$
$(\ell(\mathbf{v}_1), \ell(\mathbf{v}_2))$	$(+, +)$	$(+, ?)$
	Symmetric	Symmetric

More about the Betti set for $S = \langle n_1, n_2, n_3 \rangle$

The table continues with the nonsymmetric case (three Betti elements).

#Betti(S)	3
$\langle n_1, n_2, n_3 \rangle$	$\langle n_1, n_2, n_3 \rangle$
Betti(S)	$\{c_1 n_1, c_2 n_2, c_3 n_3\}$
Z($betti_1$)	$\{(c_1, 0, 0), (0, r_{12}, r_{13})\}$
	$c_1 > r_{12} + r_{13}$
Z($betti_2$)	$\{(0, c_2, 0), (r_{21}, 0, r_{23})\}$
Z($betti_3$)	$\{(0, 0, c_3), (r_{31}, r_{32}, 0)\}$
	$c_3 < r_{31} + r_{32}$
$\mathbf{v}_1$	$(c_1, -r_{12}, -r_{13}) = (+, -, -)$
$\mathbf{v}_2$	$(r_{31}, r_{32}, -c_3) = (+, +, -)$
$(\ell(\mathbf{v}_1), \ell(\mathbf{v}_2))$	$(+, +)$
	Non-symmetric

To unify the notation, we consider

$$\sigma = \mathrm{sg}(\ell(\mathbf{v}_2))$$

The idea

For any $x \in \{1, \ldots, \max\{\delta_1, \delta_2\}\}$ we consider the following coordinates with respect to δ_1, δ_2

$x = (x_1, x_2)$	$x = x_1\delta_1 + x_2\delta_2$ with $-\delta_1 < x_2 \leq 0 < x_1 \leq \delta_2$	$\mathbf{v}_x = x_1\mathbf{v}_1 + \sigma x_2\mathbf{v}_2$
$x = (x'_1, x'_2)$	$x = x'_1\delta_1 + x'_2\delta_2$ with $-\delta_2 < x'_1 \leq 0 < x'_2 \leq \delta_1$	$\mathbf{v}'_x = x'_1\mathbf{v}_1 + \sigma x'_2\mathbf{v}_2$

Observe that $\ell(\mathbf{v}_x) = \ell(\mathbf{v}'_x) = x$. And the signs of these vectors are

Symmetric case			Non symmetric case		
σ	$\mathbf{v}_x$	$\mathbf{v}'_x$	delta	$\mathbf{v}_x$	$\mathbf{v}'_x$
1	$(?, -, +)$	$(?, +, -)$	$\delta_1 > \delta_2$	$(?, +, -)$	$(?, -, +)$
-1	$(+, ?, -)$	$(-, ?, +)$	$\delta_2 > \delta_1$	$(-, +, ?)$	$(+, -, ?)$

An example

Let $S = \langle 2015, 7124, 84940 \rangle$

$\mathbf{v}_1 = (548, -155, 0)$, $\mathbf{v}_2 = (0, 155, -13)$, and so: $\delta_1 = 393$, $\delta_2 = 142$.

$\delta_1 = 393 \qquad \delta_2 = 142$

$D(\delta_1, \delta_2) =$	$\overset{(1,0)}{393}$	$\overset{(1,-1)}{251}$	$\overset{(1,-2)}{109}$	
$D(\delta_2, \delta_3) =$	$\overset{(0,1)}{142}$	$\overset{(-1,3)}{33}$		
$D(\delta_3, \delta_4) =$	$\overset{(1,-2)}{109}$	$\overset{(2,-5)}{76}$	$\overset{(3,-8)}{43}$	$\overset{(4,-11)}{10}$
$D(\delta_4, \delta_5) =$	$\overset{(-1,3)}{33}$	$\overset{(-5,14)}{23}$	$\overset{(-9,25)}{13}$	$\overset{(-13,36)}{3}$
$D(\delta_5, \delta_6) =$	$\overset{(4,-11)}{10}$	$\overset{(17,-47)}{7}$	$\overset{(30,-83)}{4}$	$\overset{(43,-119)}{1}$
$D(\delta_6, \delta_7) =$	$\overset{(-13,36)}{3}$	$\overset{(-56,155)}{2}$	$\overset{(-99,274)}{1}$	$\overset{(-142,393)}{0}$

$\mathrm{Euc}(\delta_1, \delta_2) = \{1, 2, 3, 4, 7, 10, 13, 23, 33, 43, 76, 109, 142, 251, 393\}$.

The same example with vectors

Recall that $S = \langle 2015, 7124, 84940\rangle$

$\mathbf{v}_1 = (548, -155, 0)$, $\mathbf{v}_2 = (0, 155, -13)$, and so: $\delta_1 = 393$, $\delta_2 = 142$.

(548,−155,0) 393	(548,−310,13) 251	(548,−465,26) 109	
(0,155,−13) 142	(−548,620,−39) 33		
(548,−465,26) 109	(1096,−1085,65) 76	(1644,−1705,104) 43	(2192,−2325,143) 10
(−548,620,−39) 33	(−2740,2945,−182) 23	(−4932,5270,−325) 13	(−7124,7595,−468) 3
(2192,−2325,143) 10	(9316,−9920,611) 7	(16440,−17515,1079) 4	(23564,−25110,1547) 1
(−7124,7595,−468) 3	(−30688,32705,−2015) 2	(−54252,57815,−3562) 1	(−77816,82925,−5109) 0

$\Delta(S) = \{1, 2, 3, 4, 7, 10, 13, 23, 33, 43, 76, 109, 142, 251, 393\}$.

The inclusion $\mathrm{Euc}(\delta_1, \delta_2) \subseteq \Delta(S)$

In the above example take, for instance, $43 \in \mathrm{Euc}(\delta_1, \delta_2)$:

$$\mathbf{v}_{43} = (1644, -1705, 104)$$

Then, we consider $1705 \cdot n_2 \in S = \langle 2015, 7124, 84940 \rangle$,

to obtain that: $(1644, 0, 104)$ and $(0, 1705, 0)$ are two factorizations of $1705 \cdot n_2$ with difference of lengths equal to 43.

Remain to prove

that there is no other factorization of the element with length between them.

$$\ell(0, 1705, 0) = 1705 < 1748 = \ell(1644, 0, 104)$$

Big problem!! All these element have same length: $\ell(\mathbf{v}) = 43$

$\mathbf{v} = \mathbf{v}_{43} + r \cdot \mathbf{v}_0$ with $r \in \mathbb{Z}$

The inclusion $\Delta(S) \subseteq \mathrm{Euc}(\delta_1, \delta_2)$ The symmetric case

Suppose $s \in S$, $\mathbf{z}$ and $\mathbf{z}'$ in $\mathsf{Z}(s)$ with $\ell(\mathbf{z}) - \ell(\mathbf{z}') \notin \mathrm{Euc}(\delta_1, \delta_2)$. We argue as follow:

- We need to find another factorization $\mathbf{z}'' \in \mathsf{Z}(s)$ such that $\ell(\mathbf{z}') < \ell(\mathbf{z}'') < \ell(\mathbf{z})$.
- Take $x = \ell(\mathbf{z} - \mathbf{z}')$, and consider d maximum in $\mathrm{Euc}(\delta_1, \delta_2)$ such that $0 < d < x$.
- Then, choose $\mathbf{v}_x$ or $\mathbf{v}'_x$ in M_S, depending on the signs of $\mathbf{z} - \mathbf{z}'$. And look for $\mathbf{v}_d \in M_S$. Actually, this $\mathbf{v}_d$ is the element to choose, commented in the last slide.
- We always have that $\ell(\mathbf{z}') < \ell(\mathbf{v}_d + \mathbf{z}') < \ell(\mathbf{z})$ and $\ell(\mathbf{z}') < \ell(\mathbf{z} - \mathbf{v}_d) < \ell(\mathbf{z})$.
- But can happen that $\mathbf{v}_d + \mathbf{z}'$, $\mathbf{z} - \mathbf{v}_d$, have some coordinate smaller than zero.
- Controlling two coordinates of $\mathbf{v}_d$, and $\mathbf{v}_x$ or $\mathbf{v}'_x$, we can assure that one of the $\mathbf{v}_d + \mathbf{z}'$ or $\mathbf{z} - \mathbf{v}_d$ is a factorization of s.
- Is important to say that the element d will be different depending on $\mathbf{v}_x$ or $\mathbf{v}'_x$.

The inclusion $\Delta(S) \subseteq \mathrm{Euc}(\delta_1, \delta_2)$ The non-symmetric case

The above argument don't work for the non-symmetric case.

- Here, we need to argue with the couples (x_1, x_2) or (x'_1, x'_2) respectively. Looking for special couples called irreducible on the role of the element d.
- Working with positive or negative components of the vector associated to this irreducible couple, in a similar way as above, we can find the desired factorization.
- Later, we need to relate these irreducible couples with the Euclid's set.

Higher dimensions

The Colton and Kaplan's example

$$S = \langle 14, 29, 30, 32, 36 \rangle$$

$$\Delta(S) = \{1, 4\}$$

If we apply our results, necessarily $\{2, 3\}$ have to belongs to the Delta set of $\langle 14, 29, 30, 32, 36 \rangle$!!!

THANKS FOR YOUR ATTENTION!!