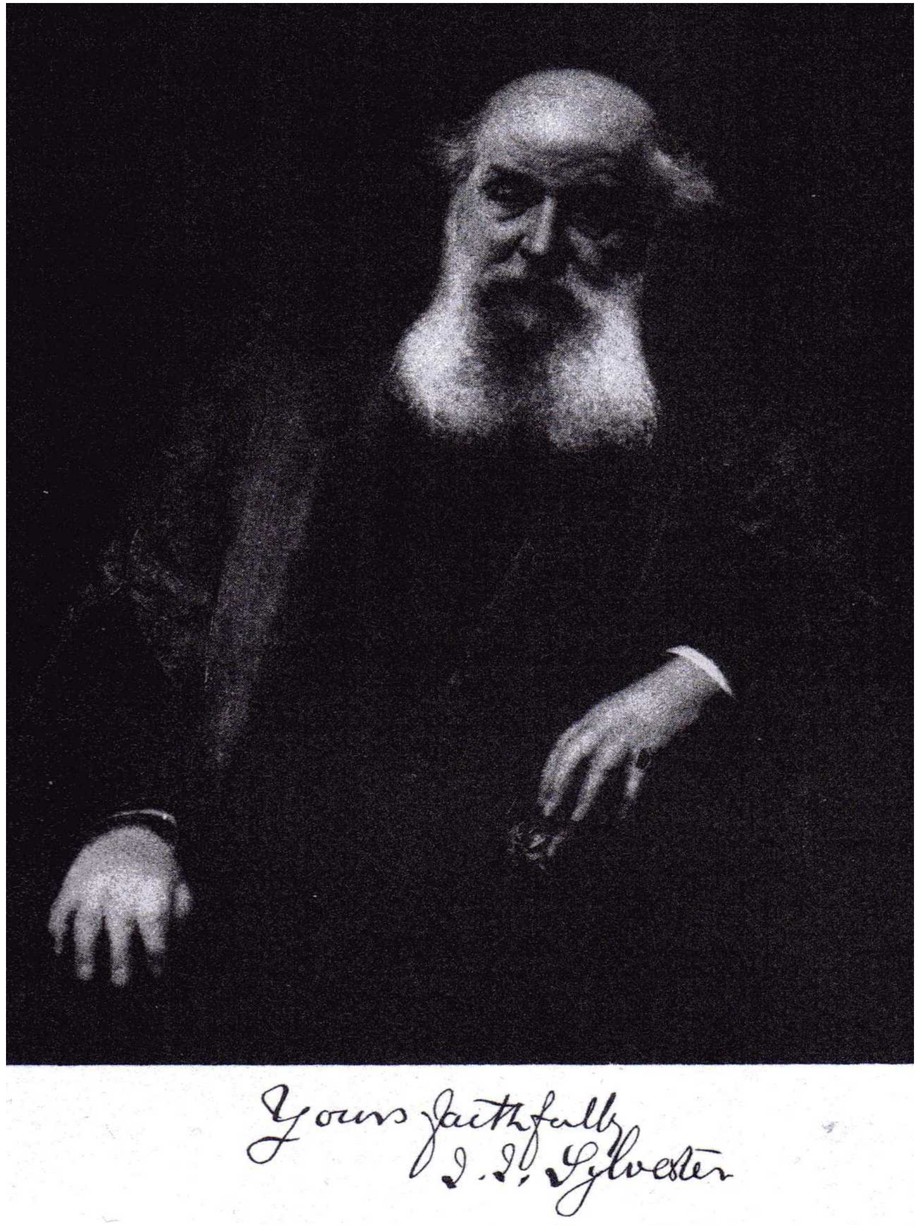
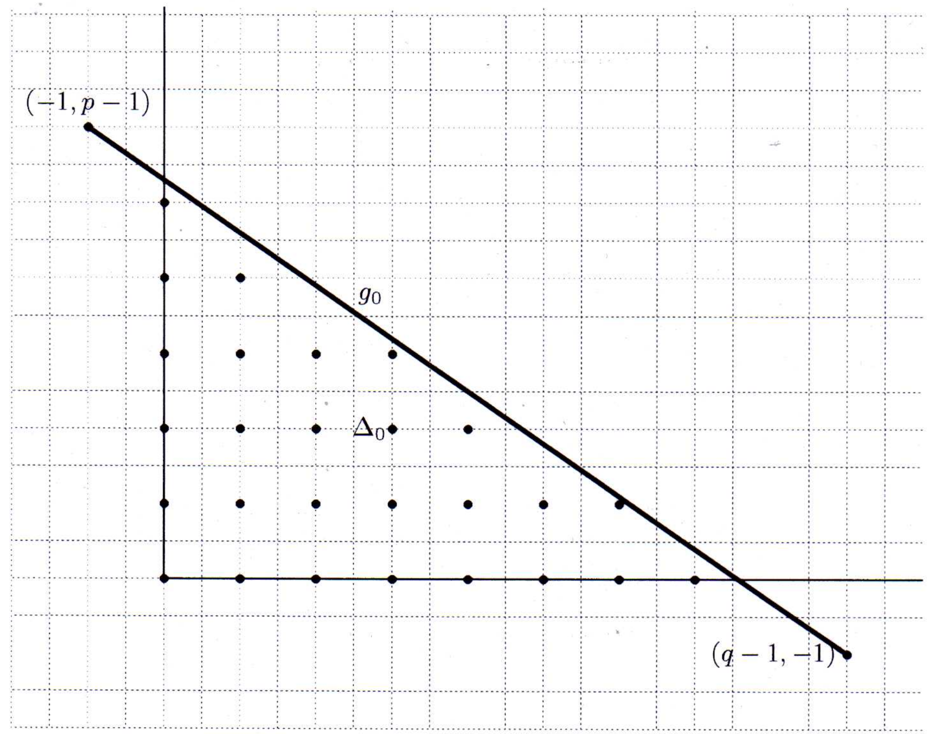
Numerical semigroups with nice properties

J. J. Sylvester (1814-1897)



Sylvester studied in 1882 the numerical semigroups generated by two coprime numbers p and q with $2 \leq p < q$. They have finitely many gaps which correspond uniquely to the lattice points in the triangle Δ_0 below.



The map

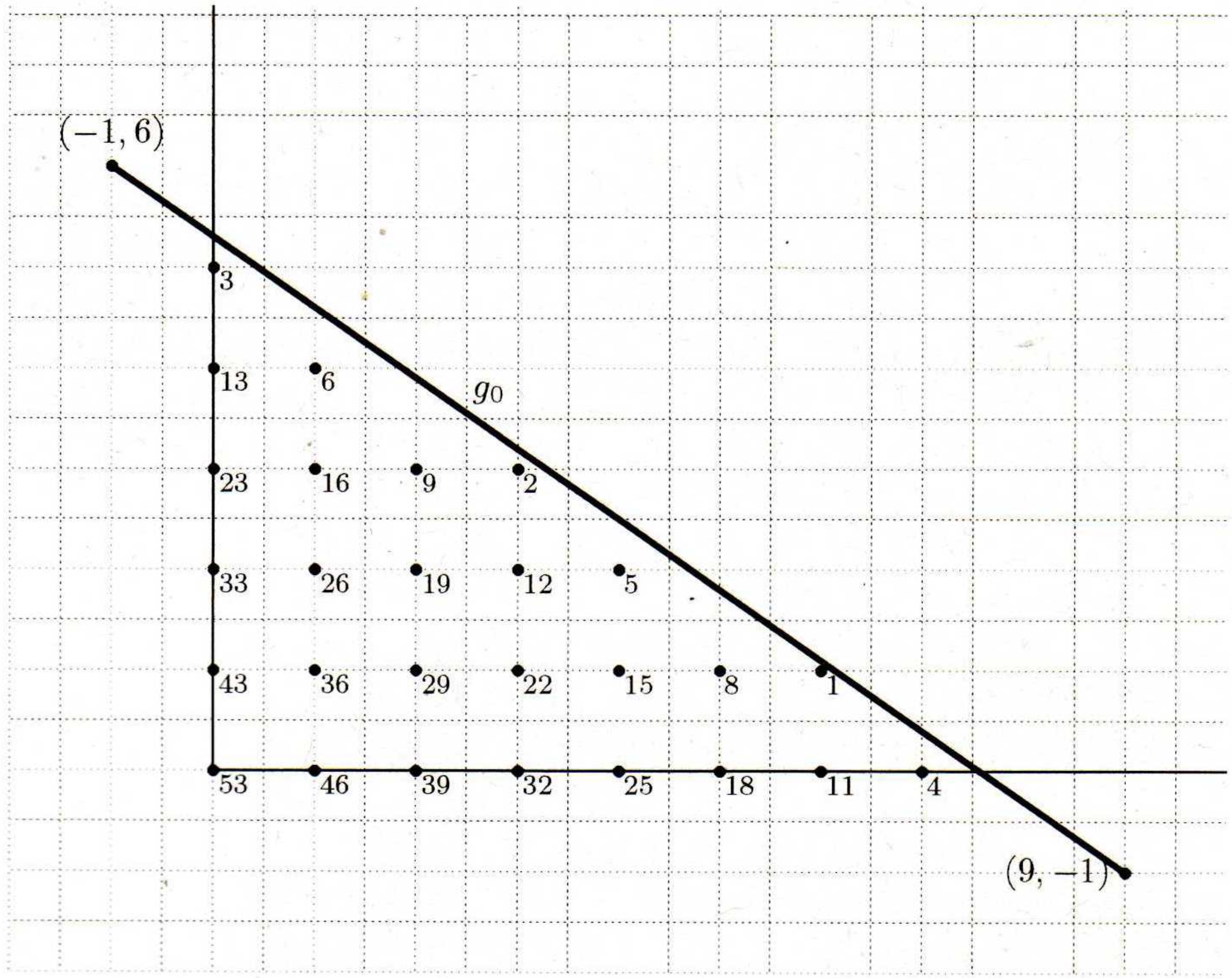
$$\gamma : \mathbb{Z}^2 \to \mathbb{Z} \quad ((a.b) \mapsto pq - (a+1)p - (b+1)q)$$

induces a bijection

$$\Delta_0 \cap \mathbb{N}^2 \to \mathbb{N} \setminus < p, q >$$

i.e.

$\gamma(\Delta_0 \cap \mathbb{N}^2) =$ set of gaps of $< p, q >$
$\gamma(0,0) =$ Frobenius number of $< p, q >$



$p = 7, q = 10$
Frobenius number=53

In the following let $3 \leq p$. We are interested in the set $\mathbf{R}(p,q)$ of all numerical semigroups H with

$$<p,q>\subset H \subset <p,q,r>,$$

where

$$r := \begin{cases} \frac{p}{2} & p \text{ even} \\ \frac{q}{2} & q \text{ even} \\ \frac{p+q}{2} & p \text{ and } q \text{ odd} \end{cases}.$$

In the terminology of Rosales and Garcia-Sanchez we have

$$<p,q,r>=\frac{<p,q>}{2}.$$

For the semigroups in $\mathbf{R}(p,q)$ many questions which are difficult to answer for arbitrary numerical semigroups have nice and easy answers. In the following examples the semigroups are given by their minimal generators, and the lists contain some of their invariants.

Examples

In the following lists g denotes the genus, F the Frobenius number, t the type and d the deviation of the semigroups. Pf stands for the Pseudo-Frobenius numbers different from F.

$\mathbf{R}(5,8), |\mathbf{R}(5,8)| = \binom{6}{2} = 15$

H	g	F	t	d	PF	
1. $< 5, 8 >$	14	27	1	0		symmetric
2. $< 4, 5 >$	6	11	1	0		symmetric
3. $< 5, 8, 9 >$	8	11	2	1	12	
4. $< 5, 8, 12 >$	10	19	1	0		symmetric
5. $< 5, 8, 14 >$	10	17	2	1	11	
6. $< 5, 8, 17 >$	11	19	2	1	12	
7. $< 5, 8, 19 >$	12	22	2	1	11	pseudo-symm.
8. $< 5, 8, 22 >$	12	19	2	1	17	
9. $< 5, 8, 27 >$	13	22	2	1	19	
10. $< 5, 8, 9, 12 >$	7	11	3	3	4,7	
11. $< 5, 8, 12, 14 >$	8	11	3	3	7,9	
12. $< 5, 8, 12, 19 >$	9	14	3	3	7,11	
13. $< 5, 8, 14, 17 >$	9	12	3	3	9,11	
14. $< 5, 8, 17, 19 >$	10	14	3	3	11,12	
15. $< 5, 8, 19, 22 >$	11	17	3	3	11,14	

$\mathbf{R}(7,10)$, $|\mathbf{R}(7,10)| = \binom{8}{3} = 56$

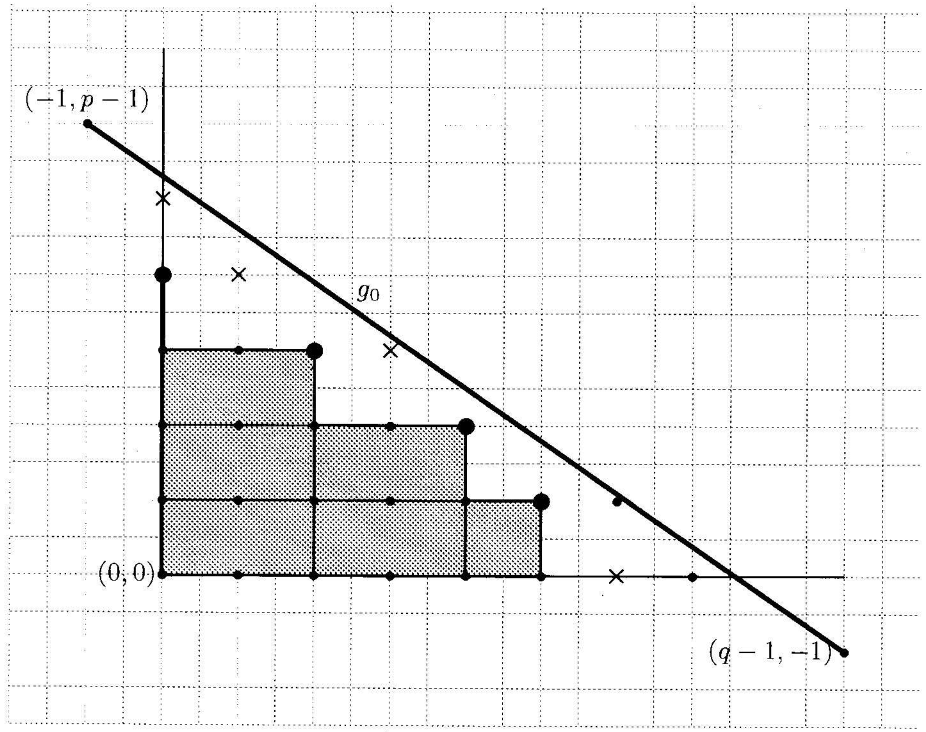
1. $< 7, 10 >$	27	53	1	0		symmetric
2. $< 5, 7 >$	12	23	1	0		symmetric
3. $< 7, 10, 12 >$	15	25	2	1	23	
4. $< 7, 10, 15 >$	17	33	1	0		symmetric
5. $< 7, 10, 19 >$	18	32	2	1	23	
6. $< 7, 10, 22 >$	19	33	2	1	25	
7. $< 7, 10, 25 >$	22	43	1	0		symmetric
8. $< 7, 10, 26 >$	21	39	2	1	23	
9. $< 7, 10, 29 >$	21	33	2	1	32	
10. $< 7, 10, 32 >$	23	43	2	1	25	pseudosymm.
11. $< 7, 10, 33 >$	24	46	2	1	23	
12. $< 7, 10, 36 >$	23	39	2	1	33	
13. $< 7, 10, 39 >$	24	43	2	1	32	
14. $< 7, 10, 43 >$	25	46	2	1	33	
15. $< 7, 10, 46 >$	25	43	2	1	39	
16. $< 7, 10, 53 >$	26	46	2	1	43	
17. $< 7, 10, 12, 15 >$	13	23	3	3	5, 18	
18. $< 7, 10, 12, 25 >$	14	23	3	3	15, 18	
19. $< 7, 10, 15, 19 >$	14	23	3	3	12, 18	
20. $< 7, 10, 15, 26 >$	15	23	3	3	18, 19	
21. $< 7, 10, 15, 33 >$	16	26	3	3	18, 23	
22. $< 7, 10, 19, 22 >$	16	25	3	3	12, 23	
23. $< 7, 10, 19, 25 >$	16	23	3	3	18, 22	
24. $< 7, 10, 19, 32 >$	17	25	3	3	22, 23	
25. $< 7, 10, 22, 25 >$	18	33	3	3	15, 18	
26. $< 7, 10, 22, 26 >$	17	25	3	3	17, 23	
27. $< 7, 10, 22, 33 >$	18	26	3	3	23, 25	
28. $< 7, 10, 25, 26 >$	18	29	3	3	18, 23	
29. $< 7, 10, 25, 29 >$	19	33	3	3	18, 22	
30. $< 7, 10, 25, 33 >$	20	36	3	3	18, 23	

31. $< 7, 10, 25, 36 >$	20	33	3	3	18, 29
32. $< 7, 10, 25, 43 >$	21	36	3	3	18, 33
33. $< 7, 10, 26, 29 >$	19	32	3	3	19, 23
34. $< 7, 10, 26, 32 >$	19	29	3	3	23, 25
35. $< 7, 10, 26, 39 >$	20	32	3	3	23, 29
36. $< 7, 10, 29, 32 >$	20	33	3	3	22, 25
37. $< 7, 10, 29, 33 >$	20	32	3	3	23, 26
38. $< 7, 10, 32, 33 >$	21	36	3	3	23, 25
39. $< 7, 10, 32, 36 >$	21	33	3	3	25, 29
40. $< 7, 10, 32, 43 >$	22	36	3	3	25, 33
41. $< 7, 10, 33, 36 >$	22	39	3	3	23, 26
42. $< 7, 10, 33, 39 >$	22	36	3	3	23, 32
43. $< 7, 10, 33, 46 >$	23	38	3	3	23, 36
44. $< 7, 10, 36, 39 >$	22	33	3	3	29, 32
45. $< 7, 10, 39, 43 >$	23	36	3	3	32, 33
46. $< 7, 10, 43, 46 >$	24	39	3	3	33, 36

47. $< 7, 10, 19, 22, 25 >$	15	23	4	6	12, 15, 18
48. $< 7, 10, 22, 25, 26 >$	16	23	4	6	15, 18, 19
49. $< 7, 10, 22, 25, 33 >$	17	26	4	6	15, 18, 23
50. $< 7, 10, 25, 26, 29 >$	17	23	4	6	18, 19, 22
51. $< 7, 10, 25, 29, 33 >$	18	26	4	6	18, 22, 23
52. $< 7, 10, 25, 33, 36 >$	19	29	4	6	18, 23, 26
53. $< 7, 10, 26, 29, 32 >$	18	25	4	6	19, 22, 23
54. $< 7, 10, 29, 33, 36 >$	19	26	4	6	22, 23, 25
55. $< 7, 10, 32, 33, 36 >$	20	29	4	6	23, 25, 26
56. $< 7, 10, 33, 36, 39 >$	21	32	4	6	23, 26, 29

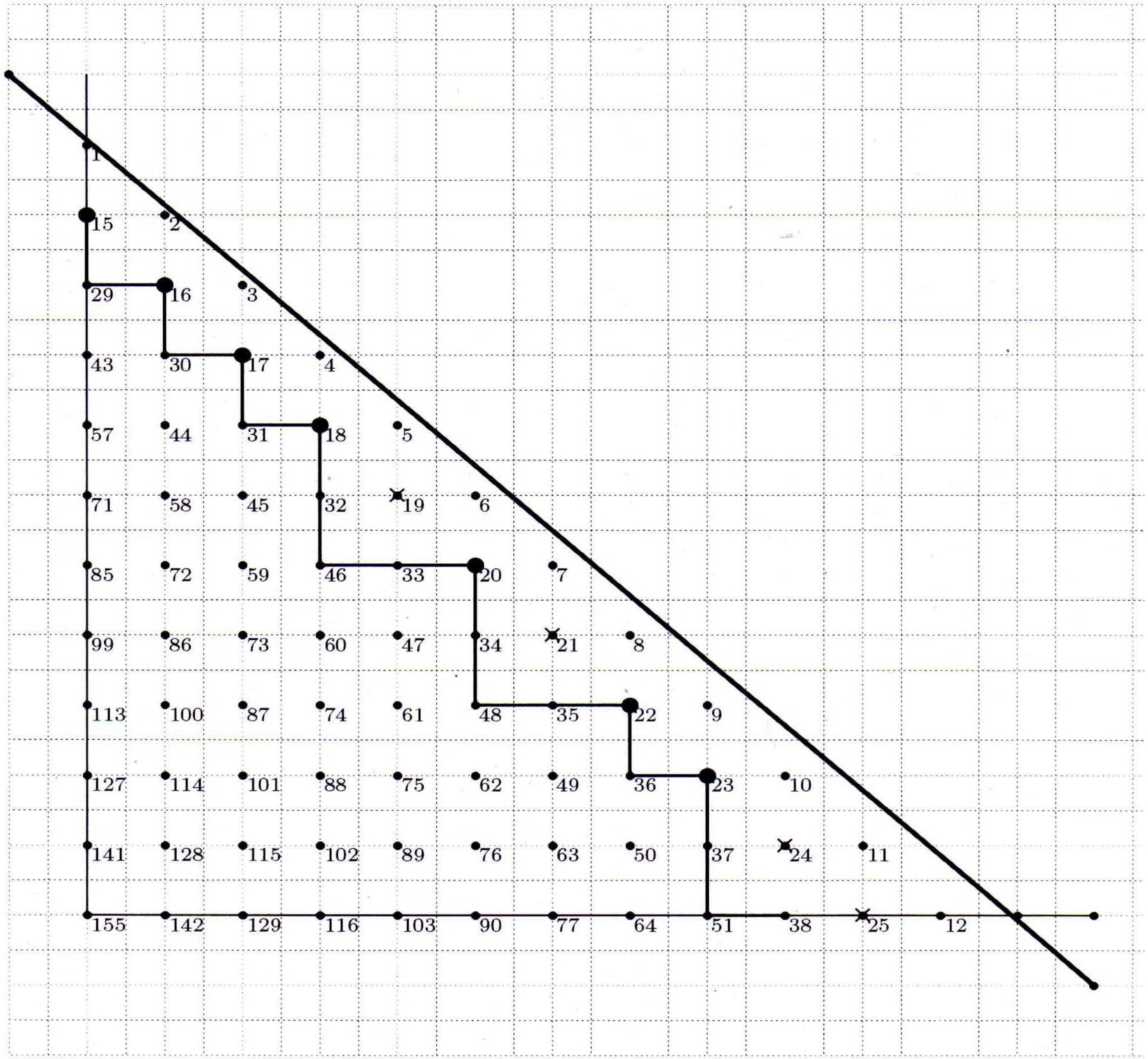
$|\mathbf{R}(73, 83)| = \binom{77}{36} = 35000417292158999098110$

Semigroups H containing p and q are obtained from $< p, q >$ by closing some of its gaps. The closed gaps correspond to the lattice points in the area bounded by the coordinate axis and a lattice path having only left and downward steps (the shaded region in the figure below).



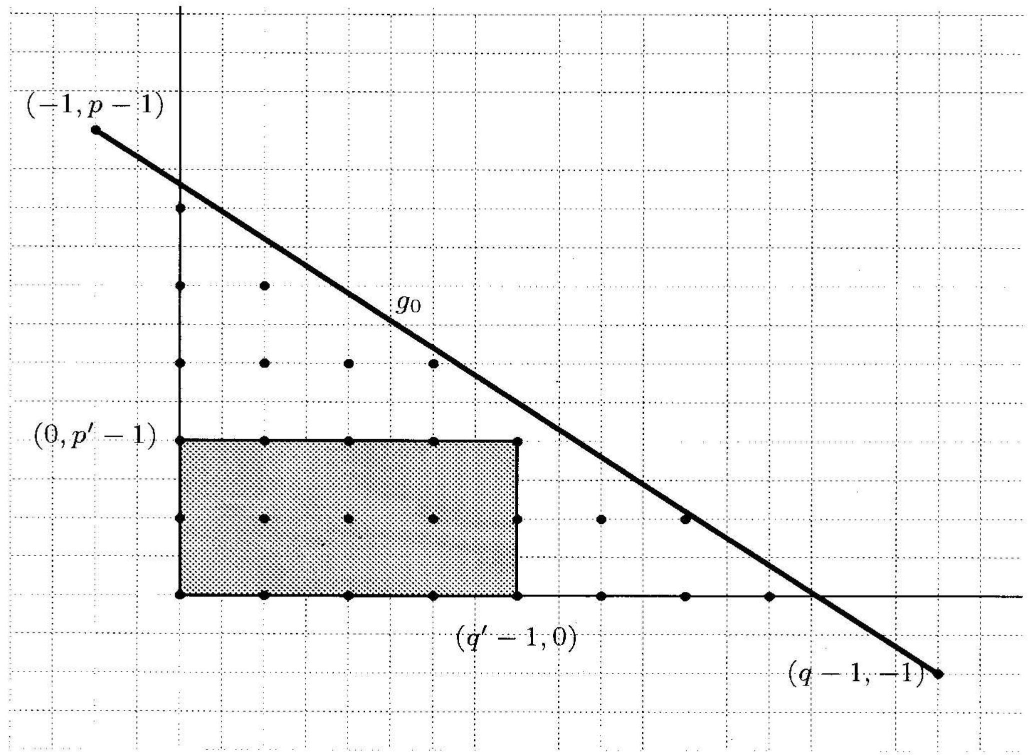
The big dots are called the **corners** of the lattice path or the semigroup, the crossed points the **interesting points** of H.

The γ-values of the corners together with p and q form a system of generators of H, those of the interesting points are candidates for the pseudo-Frobenius numbers of H. The interesting point on the left-most parallel to g_0 gives the Frobenius number.



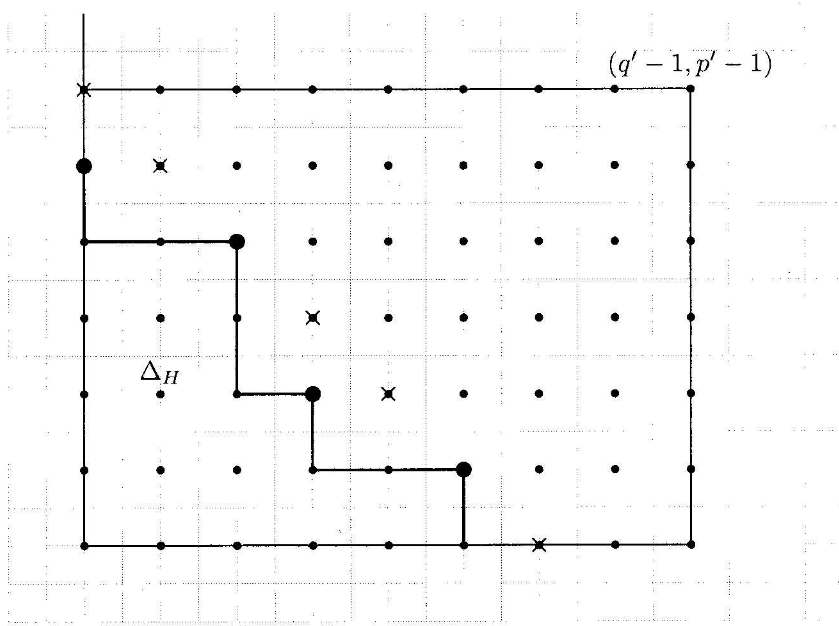
$H = < 13, 14, 15, 16, 17, 18, 20, 22, 23 >,\ p = 13, q = 14$

$PF = \{19, 21, 24, 25\}$



$p' := \lfloor \frac{p}{2} \rfloor, q' := \lfloor \frac{q}{2} \rfloor$

The shaded rectangle defines $\frac{<p,q>}{2}$, the lattice paths in the rectangle are in one-to-one correspondence to the semigroups in $\mathbf{R}(p,q)$.



(q′ − 1, p′ − 1)
Δ_H

The γ-values of the corners are minimal generators of H. Together with p and q they form a minimal system of generators of H (except when p or q is even, which occurs only for $H =< p/2, q >$ or $H =< p, q/2 >$).
(Geometrical illustration of numerical semigroups and of some of their invariants. Semigroup Forum 89 (2014) 664-691)

If $\mathbf{R}(p, q)$ with $\mathrm{edim}(H) = 3$ is not symmetric or $\mathrm{edim}(H) \geq 4$, then the γ-values of the interesting points $\times$ are pseudo-Frobenius numbers. Therefore

Theorem $t(H) = \mathrm{edim}(H) - 1$ for these semigroups H and H is uniquely determind by its pseudo-Frobenius numbers.

In R. Fröberg, C. Gottlieb and H. Häggkvist, On numerical semigroups. Semigroup Forum 35 (1987) 63-83 it was shown that Wilf's question has a positive answer if $t(H) < \mathrm{edim}(H)$. This is the case in $\mathbf{R}(p, q)$.

In the rectangle there are $\binom{p'+q'}{p'}$ lattice paths, hence

$$|\mathbf{R}(p,q)| = \binom{p'+q'}{p'}.$$

For each $s \in \{1,\ldots,p'\}$ there are exactly

$$\binom{p'}{s} \cdot \binom{q'}{s}$$

lattice paths with s corners and therefore as many $H \in \mathbf{R}(p,q)$ with $\operatorname{edim}(H) = s+2$ (one less for $s=1$ if p or q is even).

The maximal embedding dimension of an $H \in \mathbf{R}(p,q)$ is $p'+2$ and there are $\binom{q'}{p'}$ semigroups with this embedding dimension.

Let K be a field and $K[[H]] = K[[\{t^h\}_{h\in H}]]$ the completed semigroup algebra of a numerical semigroup H over K and let $\{h_1, \dots, h_n\}$ be a system of generators of H. If I is the kernel of the K-homomorphism

$$K[[X_1, \dots, X_n]] \to K[[H]] \ (X_i \mapsto t^{h_i})$$

and $\mu(I)$ its minimal number of generators, then

$$d(H) := \mu(I) - (n-1)$$

is called the *deviation* of H (or of the affine monomial curve defined by the t^{h_i}). It does not depend on the choice of the system of generators of H.

Theorem. For $H \in \mathbf{R}(p,q)$ with $\operatorname{edim}(H) \geq 4$ let $(a_i, b_i) \in \mathbb{N}^2$ $(i = 1, \ldots, s := \operatorname{edim}(H) - 2)$ be the corners of the lattice path of H. Then the relation ideal I of $K[[H]]$ in $K[[X, Y, X_1, \ldots, X_s]]$ for $(X \mapsto t^p, Y \mapsto t^q, X_i \mapsto t^{\gamma(a_i,b_i)})$ has the following minimal system of generators

$$\{X_i X_j - X^{q-a_i-a_j-2} Y^{p-b_i-b_j-2}\}_{i,j=1,\ldots,s,i\leq j}$$

$$\cup\{Y^{b_i-b_{i+1}} X_i - X^{a_{i+1}-a_i} X_{i+1}\}_{i=1,\ldots,s-1}$$

$$\cup\{Y^{p-b_1-1} X_1 - X^{a_1+1}, Y^{b_s+1} X_s - X^{q-a_s-1}\}.$$

Note that the exponents $q - a_i - a_j - 2 \in \mathbb{N}$ and $p - b_i - b_j - 2 \in \mathbb{N}$ for $H \in \mathbf{R}(p,q)$. Counting the minimal relations we obtain

Corollary. For non-symmetric $H \in \mathbf{R}(p,q)$ with $\operatorname{edim}(H) = 3$ and all H with $\operatorname{edim}(H) \geq 4$ we have

$$d(H) = \binom{\operatorname{edim}(H) - 1}{2} = \binom{t(H)}{2}.$$

Sketch of proof

a) The polynomials belong to the relation ideal since they correspond to relations in H:

$$\gamma(a_i, b_i) + \gamma(a_j, b_j) = (q - a_i - a_j - 2)p + (p - b_i - b_j - 2)q \ (i, j = 1, \ldots, s, i \leq j)$$

$$(b_i - b_{i+1})q + \gamma(a_i, b_i) = (a_{i+1} - a_i)p + \gamma(a_{i+1}, b_{i+1}) \ (i = 1, \ldots, s - 1)$$

$$(p - b_1 - 1)q = (a_1 + 1)p + \gamma(a_1, b_1)$$

$$(b_s + 1)q + \gamma(a_s, b_s) = (q - a_s - 1)p.$$

b) If the polynomials generate I they are a minimal system of generators: When we set $X = 0$ we get the monomials

$$X_i X_j \ (i, j = 1, \dots, s, i \leq j), Y^{b_i - b_{i+1}} X_i \ (i = 1, \dots, s-1), Y^{p-b_1-1}, Y^{b_s+1} X_s$$

which are independent. By Nakayama the original polynomials are independent, too.

c) Let $I' \subset I$ be the ideal they generate. Use that

$$K[[H]] = K[[t^p, t^q]] \oplus \bigoplus_{\gamma \in \Gamma} K t^\gamma$$

where Γ is the set of closed gaps of H. With some calculation one finds that $X^q - Y^p \in I'$ and that the canonical surjection

$$K[[X, Y, X_1, \dots, X_s]]/I' \to K[[t^p, t^q]] \oplus \bigoplus_{\gamma \in \Gamma} K t^\gamma$$

is bijective, hence $I' = I$.

Corrections

Slide 16: In the theorem replace the polynomial $Y^{p-b_1-1}X_1 - X^{a_1+1}$ by

$$Y^{p-b_1-1} - X^{a_1+1}X_1.$$

Slide 18: Replace $Y^{p-b_1-1}X_1$ by Y^{p-b_1-1}.