Duality on Value Semigroups

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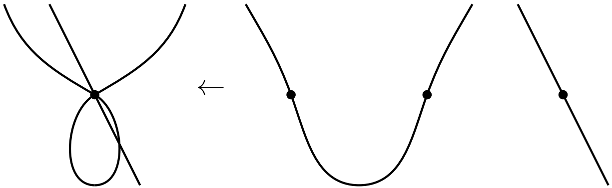
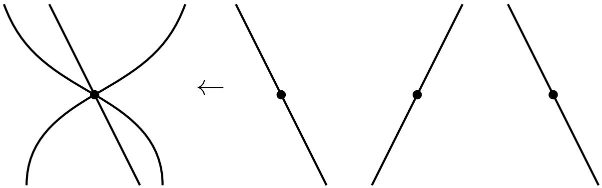




Example

Complex algebroid curve

$$\begin{aligned} R &= \mathbb{C}[[x,y]]/\langle x^3y + y^6\rangle = \mathbb{C}[[x,y]]/(\langle x^3 + y^5\rangle \cap \langle y\rangle) \\ &= \mathbb{C}[[(t_1^5, t_2), (-t_1^3, 0)]] \\ &\subset \overline{R} = \mathbb{C}[[t_1]] \times \mathbb{C}[[t_2]] = \overline{R/\langle x^3 + y^5\rangle} \times \overline{R/\langle y\rangle} \\ &\subset Q_R = \mathbb{C}[[t_1]][t_1^{-1}] \times \mathbb{C}[[t_2]][t_2^{-1}] \end{aligned}$$

Parametrization

$$x \mapsto (t_1^5, t_2), \quad y \mapsto (-t_1^3, 0)$$

Discrete valuations

$$\nu_i = \mathrm{ord}_{t_i} : \mathbb{C}[[t_i]][t_i^{-1}] \to \mathbb{Z} \cup \{\infty\}$$

Value Semigroup

Definition

R complex algebroid curve
$\rightsquigarrow$ multivaluation

$$\nu = (\nu_1, \dots, \nu_s)\colon Q_R^{\mathrm{reg}} \to \mathbb{Z}^s,$$

$\rightsquigarrow$ value semigroup of R

$$\Gamma_R = \nu(R^{\mathrm{reg}}) \subset \mathbb{N}^s.$$

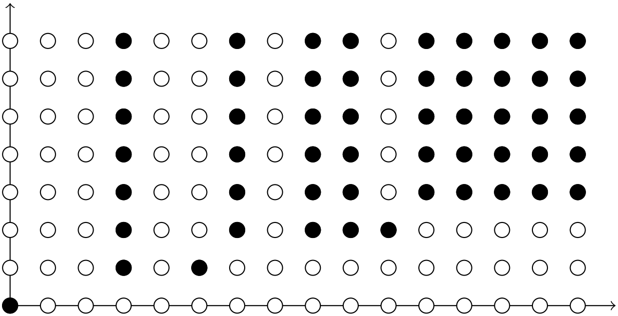
Remark

Since $\nu(1) = 0$ and $\nu(ab) = \nu(a) + \nu(b)$, Γ_R is a monoid.

Example

$R = \mathbb{C}[[x,y]]/\langle x^3y + y^6\rangle \cong \mathbb{C}[[(t_1^5, t_2), (-t_1^3, 0)]]$. Then

$$\Gamma_R = \langle (5,1), (9,2), (3,1) + \mathbb{N}\mathbf{e}_1, (15,3) + \mathbb{N}\mathbf{e}_2 \rangle$$



Fractional Ideals

Definition

- A regular fractional ideal (RFI) of R is an R-submodule $\mathcal{E} \subset Q_R$ such that $a\mathcal{E} \subset R$ for some $a \in R^{\mathrm{reg}}$ and $\mathcal{E} \cap Q_R^{\mathrm{reg}} \neq \emptyset$.
- The value semigroup ideal of $\mathcal{E}$ is

$$\Gamma_{\mathcal{E}} = \nu(\mathcal{E} \cap Q_R^{\mathrm{reg}}) \subset \mathbb{Z}^s.$$

Remark

Applying ν to $R\mathcal{E} \subset \mathcal{E}$ yields

$$\Gamma_{\mathcal{E}} + \Gamma_R \subset \Gamma_{\mathcal{E}}.$$

Definition

The conductor of R is

$$\mathcal{C}_R = R :_{Q_R} \overline{R},$$

the largest ideal of $\overline{R}$ in R.

Lemma

$$\mathcal{C}_R = t^{\gamma}\overline{R} = (t_1^{\gamma_1}, \ldots, t_s^{\gamma_s})\overline{R},$$

where

$$\gamma = \min\{\alpha \in \Gamma_R \mid \alpha + \mathbb{N}^s \subset \Gamma_R\}$$

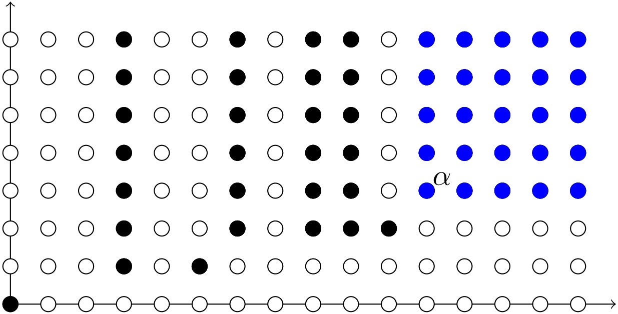
is the conductor of Γ_R.

Properties of Value Semigroups

(E0)

There is an $\alpha \in E$ such that $\alpha + \mathbb{N}^s \subset E$.

Example



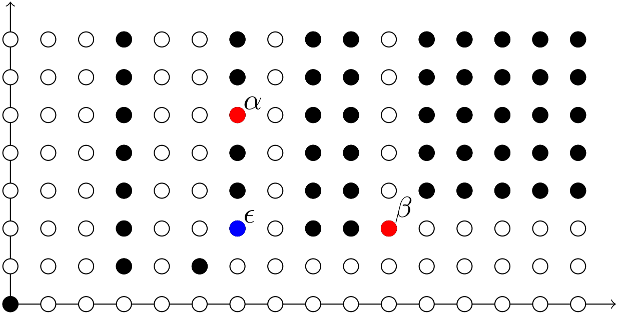
$$(t_1^{11}, t_2^{3})(\mathbb{C}[[t_1]] \times \mathbb{C}[[t_2]]) \subset R$$

Properties of Value Semigroups

(E1)

If $\alpha, \beta \in E$, then $\epsilon = \min\{\alpha, \beta\} \in E$.

Example



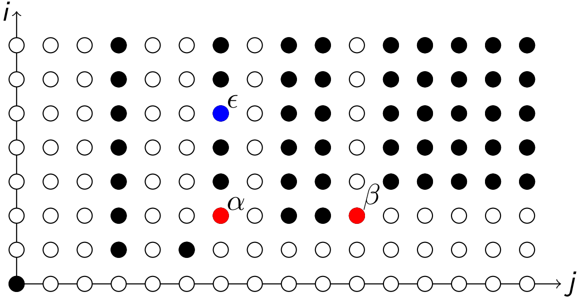
$$(t_1^{10}, t_2^{2}) + (t_1^{6} + t_1^{25}, t_2^{5}) = (t_1^{6} + t_1^{10} + t_1^{25}, t_2^{2} + t_2^{5})$$

Properties of Value Semigroups

(E2)

For any $\alpha, \beta \in E$ with $\alpha_i = \beta_i$ for some i there is ϵ in E such that $\epsilon_i > \alpha_i = \beta_i$ and $\epsilon_j \geq \min\{\alpha_j, \beta_j\}$ for all $j \neq i$ with equality if $\alpha_j \neq \beta_j$.

Example



$$(t_1^6 + t_1^{10} + t_1^{25}, t_2^2 + t_2^5) - (t_1^{10}, t_2^2) = (t_1^6 + t_1^{25}, t_2^5)$$

Good Semigroups and their Ideals

Definition

- A submonoid $S \subset \mathbb{N}^s$ with group of differences $\mathbb{Z}^s$ is called a good semigroup if (E0), (E1) and (E2) hold for S.
- A good semigroup ideal (GSI) of S is a subset $\emptyset \neq E \subset \mathbb{Z}^s$ such that
  - $E + S \subset E$ ($\rightsquigarrow$ (E0)),
  - there is an $\alpha \in S$ such that $\alpha + E \subset S$,
  - E satisfies (E1) and (E2).

Remark (Barucci, D'Anna, Fröberg)

Not any good semigroup is a value semigroup.

General algebraic hypotheses

- R one-dimensional semilocal Cohen–Macaulay ring
  $\rightsquigarrow$ there are finitely many valuations of Q_R containing R, all are discrete
- R analytically reduced $\rightsquigarrow$ (E0)
- R has large residue fields $\rightsquigarrow$ (E1)
- R residually rational $\rightsquigarrow$ (E2)

Definition

We call a one-dimensional semilocal analytically reduced and residually rational Cohen–Macaulay ring with large residue fields admissible.

Theorem

Let R be an admissible ring, $\mathcal{E}$ a RFI of R. Then:

- Γ_R *is a good semigroup.*
- $\Gamma_{\mathcal{E}}$ *is a good semigroup ideal.*
- $\Gamma_{\mathcal{E}} = \prod_{\mathfrak{m} \in \mathrm{Max}(R)} \Gamma_{\mathcal{E}_{\mathfrak{m}}}$.
- $\Gamma_{\mathcal{E}} = \Gamma_{\widehat{\mathcal{E}}}$.

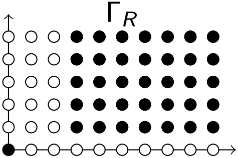
Remark

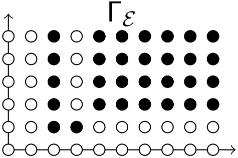
In general,

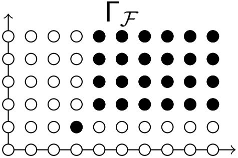
- $\Gamma_{\mathcal{E}:\mathcal{F}} \subsetneq \Gamma_{\mathcal{E}} - \Gamma_{\mathcal{F}}$,
- $\Gamma_{\mathcal{E}} + \Gamma_{\mathcal{F}} \subsetneq \Gamma_{\mathcal{E}\mathcal{F}}$,
- $\Gamma_{\mathcal{E}} - \Gamma_{\mathcal{F}}$ not GSI,
- $\Gamma_{\mathcal{E}} + \Gamma_{\mathcal{F}}$ not GSI.

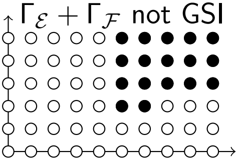
Example

$$\begin{aligned}
R &= \mathbb{C}[[(-t_1^4, t_2), (-t_1^3, 0), (0, t_2), (t_1^5, 0)]] \\
\mathcal{E} &= \langle (t_1^3, t_2), (t_1^2, 0) \rangle_R \\
\mathcal{F} &= \langle (t_1^3, t_2), (t_1^4, 0), (t_1^5, 0) \rangle_R
\end{aligned}$$





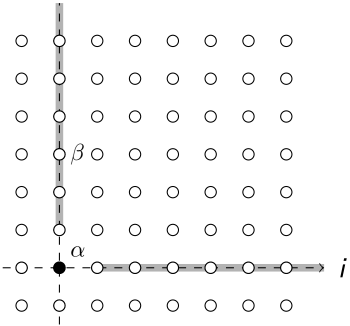




Definition (Delgado)

For $\alpha \in \mathbb{Z}^s$, consider set

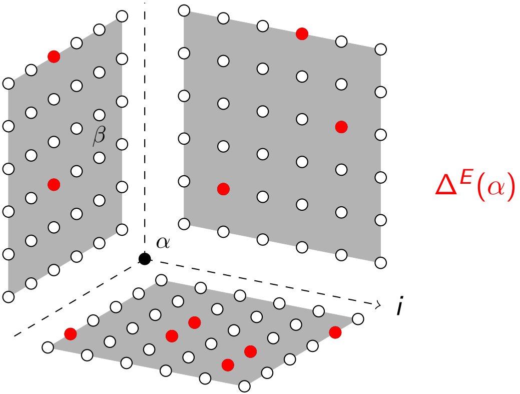
$$\Delta(\alpha) = \bigcup_{i=1}^{s} \{\beta \in \mathbb{Z}^s \mid \alpha_i = \beta_i,\ \alpha_j < \beta_j \text{ for all } j \neq i\}.$$



Definition

For $E \subset \mathbb{Z}^s$, set

$$\Delta^E(\alpha) = \Delta(\alpha) \cap E.$$



Definition

The conductor of a good semigroup ideal E is

$$\gamma_E = \min\{\alpha \in E \mid \alpha + \mathbb{N}^s \subset E\},$$

and we set $\tau = \gamma_S - \mathbf{1}$.

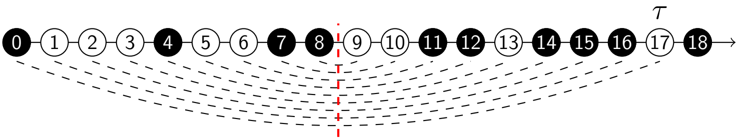
Theorem (Delgado / Campillo, Delgado, Kiyek)

Let R be a local admissible ring. Then R is Gorenstein if and only if $\Gamma_R = \{\alpha \in \mathbb{Z}^s \mid \Delta^S(\tau - \alpha) = \emptyset\}$ (Γ_R symmetric).

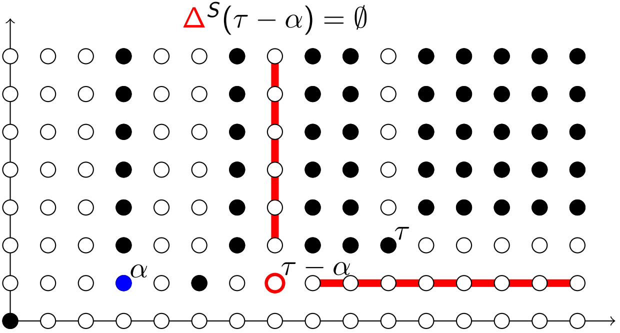
Example

Irreducible plane curve

$$R = \mathbb{C}[[x, y]]/\langle x^7 - y^4 \rangle \cong \mathbb{C}[[t^4, t^7]]$$

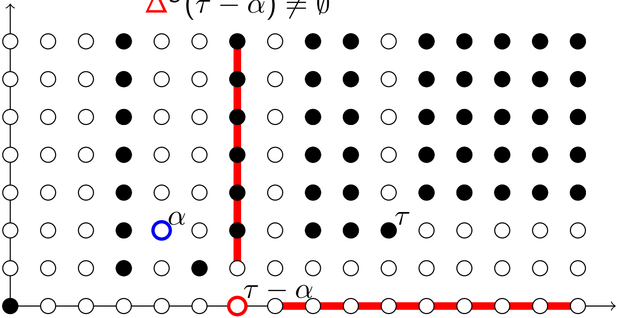


Example



Example

$\Delta^S(\tau - \alpha) \neq \emptyset$



Definition

- A RFI $\mathcal{K}$ is canonical if $\mathcal{K} : (\mathcal{K} : \mathcal{E}) = \mathcal{E}$ for all RFI $\mathcal{E}$.
- R is Gorenstein if R is a canonical ideal.

Definition (D'Anna)

The canonical semigroup ideal of a good semigroup S is

$$K_S^0 = \{\alpha \in \mathbb{Z}^s \mid \Delta^S(\tau - \alpha) = \emptyset\}.$$

Remark

R is Gorenstein if and only if $\Gamma_R = K^0_{\Gamma_R}$.

Theorem (D'Anna)

Let R be local and $\mathcal{K}$ a RFI such that $R \subset \mathcal{K} \subset \overline{R}$. Then

$$\mathcal{K} \text{ canonical} \Longleftrightarrow \Gamma_{\mathcal{K}} = K^0_{\Gamma_R}.$$

Theorem (Pol)

Let R be a Gorenstein algebroid curve and $\mathcal{E}$ a RFI. Then

$$\begin{aligned}\Gamma_{R:\mathcal{E}} &= \{\alpha \in \mathbb{Z}^s \mid \Delta^{\Gamma_{\mathcal{E}}}(\tau - \alpha) = \emptyset\} \\ &= \Gamma_R - \Gamma_{\mathcal{E}}.\end{aligned}$$

Definition (KTS)

Let S be a good semigroup. We call K a canonical semigroup ideal if

- K GSI
- If E GSI with $K \subset E$ and $\gamma_K = \gamma_E$, then $K = E$.

Theorem (KTS)

The following are equivalent:

- *K is a canonical semigroup ideal.*
- *$\alpha + K = K_S^0$ for some $\alpha \in \mathbb{Z}^s$.*
- *$K - (K - E) = E$ for all GSI E.*

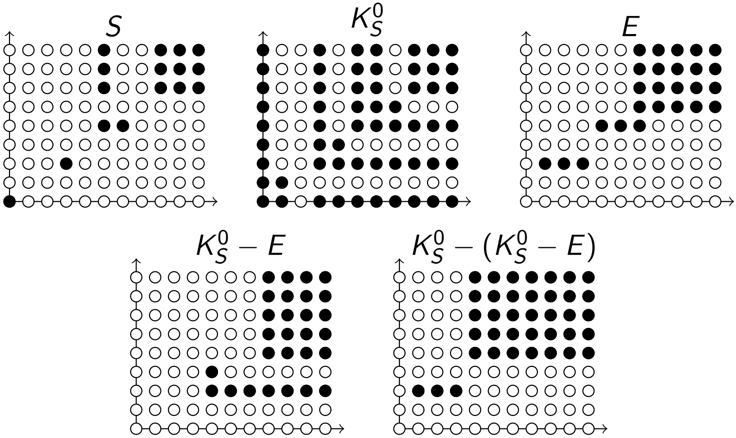
If these are satisfied, then

$$K - E = \{\beta \in \mathbb{Z}^s \mid \Delta^E(\tau - \beta) = \emptyset\} + \alpha$$

is a GSI.

Example

E is (E1) but not (E2), $K_S^0 - E$ not GSI, $E \subsetneq K_S^0 - (K_S^0 - E)$.



Main Result

Theorem (KTS)

Let R be an admissible ring, $\mathcal{K}$ a RFI.

- *$\mathcal{K}$ is canonical if and only if $\Gamma_{\mathcal{K}}$ is canonical.*
- *If $\mathcal{K}$ is canonical, then*

$$
\begin{array}{ccc}
\{RFI \text{ of } R\} & \xrightarrow{\mathcal{E} \mapsto \mathcal{K}:\mathcal{E}} & \{RFI \text{ of } R\} \\
{\scriptstyle \mathcal{E} \mapsto \Gamma_{\mathcal{E}}} \big\downarrow & \circlearrowleft & \big\downarrow {\scriptstyle \mathcal{E} \mapsto \Gamma_{\mathcal{E}}} \\
\{GSI \text{ of } \Gamma_R\} & \xrightarrow{E \mapsto \Gamma_{\mathcal{K}} - E} & \{GSI \text{ of } \Gamma_R\}
\end{array}
$$

Reference

[KTS] Philipp Korell, Laura Tozzo, and Mathias Schulze: "Duality on value semigroups", arXiv 1510.04072 (2015).