Arf Numerical Semigroups with Multiplicity ≤ 6

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$\mathbf{N}_0$: the set of nonnegative integers, S : a numerical semigroup.

$\{a_1, a_2, \ldots, a_e\}$: *minimal set of generators* for S with $a_1 < a_2 < \ldots < a_e$.

$a_1 = m = m(S)$: *multiplicity* of S. $a_2 = R = R(S)$: *ratio* of S.

$e = e(S)$: *embedding dimension* of S.

S is said to be of *maximal embedding dimension if* $e(S) = m(S)$.

$F = F(S)$: *Frobenius number* of S.

$C = C(S)$: *Conductor* of S. $C(S) = F(S)+1$; $C(\mathbf{N}_0) = 0$ and $C(S) \geq 2$ iff $S \neq \mathbf{N}_0$.

$S = \{s_0 = 0, s_1, s_2, \ldots, s_{n-1}, s_n = C \longrightarrow\}$, $s_0 = 0 < s_1 < s_2 < \ldots < s_{n-1} < s_n = C$

$\mathbf{N_0} \setminus S = \{$ *gaps* of $S\}$, $|\mathbf{N_0} \setminus S| = G = G(S)$: *Genus* of S

$a \in S\setminus\{0\}$. $Ap(S, a) = \{s \in S : s - a$ is not in $S\}$ is the *Apery set* of S with respect to a.

$Ap(S, a) = \{0, w(1), \ldots, w(a-1)\}$, $w(i)$: the least element of S such that $w(i) \equiv i \ (mod\ a)$.

$S = \langle a, w(1), \ldots, w(a-1) \rangle$, $F(S) = max(Ap(S, a)) - a$.

S : Arf numerical semigroup. $x, y, z \in S; x \geq y \geq z \Rightarrow x + y - z \in S$.

Every Arf numerical semigroup is of maximal embedding dimension: $e(S) = m(S)$

$\{m, w(1), \ldots, w(m-1)\}$ is the minimal set of generators of S if S is Arf.

The largest element of the set $\{m, w(1), \ldots, w(m-1)\}$ is $F + m = C + m - 1$.

Lemma 1. *A numerical semigroup S is an Arf numerical semigroup if and only if* $2x - y \in S$ *for all* $x, y \in S$ *with* $x \geq y$. (Dobbs and Mathews)

David E. Dobbs and Gretchen L. Mathews, *On comparing two chains of numerical semigroups and detecting Arf semigroups, Semigroup Forum* **63** (2001), 237 - 246.

Lemma 2. *Let S be an Arf numerical semigroup and let s be any element of S. If* $s + 1 \in S$*, then* $s + k \in S$ *for all* $k \in \mathbf{N}_0$ *and thus* $C \leq s$. (Rosales, Garcia-Sanchez, Garcia-Garcia and Branco)

J. C. Rosales, P. A. Garcia-Sanchez, J. I. Garcia-Garcia, and M. B. Branco, *Arf numerical semigroups, J. of Alg.* **276** (2004), 3 - 12.

Lemma 3. *Let S be an Arf numerical semigroup with multiplicity* $m > 2$ *and conductor C. For each* $j = 2, 3, \ldots, m - 1$*, we have*

$$(i)\ w(j-1) < w(j) \Rightarrow C \leq w(j) - 1 \quad , \quad (ii)\ w(j) < w(j-1) \Rightarrow C \leq w(j-1).$$

Lemma 4. *Let S be an Arf numerical semigroup with multiplicity m and conductor C where* $C \equiv k \pmod{m}$, $k \in \{0, 2, \ldots, m-1\}$. *Then*

$$(i)\ w(1) = \begin{cases} C+1 & , k = 0 \\ C-k+m+1 & , k \neq 0 \end{cases} \quad , \quad (ii)\ w(m-1) = C-k+m-1.$$

Lemma 5. *Let S be an Arf numerical semigroup with multiplicity* $m > 2$. *For any positive integer* $k < m/2$, *we have*

$$(i)\ w(2k) \leq w(k) + k \quad , \quad (ii)\ w(m-2k) \leq w(m-k) + m - k.$$

Arf numerical semigroups with multiplicity 2 and conductor C :

$$C \text{ is even}, \; S = \langle 2, C+1 \rangle.$$

Arf numerical semigroups with multiplicity 3 and conductor C :

$C \equiv 0$ or $2 \pmod{3}$.

$$C \equiv 0 \pmod{3} \Rightarrow S = \langle 3, C+1, C+2 \rangle.$$

$$C \equiv 2 \pmod{3} \Rightarrow S = \langle 3, C, C+2 \rangle.$$

Arf numerical semigroups with multiplicity 4 and conductor C :

$C \equiv 0, 2$ or $3 \pmod{4}$.

$$C \equiv 0 \pmod{4} \Rightarrow S = \langle 4, 4t+2, C+1, C+3 \rangle, \; 1 \le t \le C/4.$$

$$C \equiv 2 \pmod{4} \Rightarrow S = \langle 4, 4t+2, C+1, C+3 \rangle, \; 1 \le t \le (C-2)/4.$$

$$C \equiv 3 \pmod{4} \Rightarrow S = \langle 4, C, C+2, C+3 \rangle.$$

Arf numerical semigroups with multiplicity 5 and conductor C :

$C \equiv 0, 2, 3$ or $4 \ (mod\ 5)$.

$$C \equiv 0 \ (mod\ 5) \Rightarrow S = \langle 5, C-2, C+1, C+2, C+4 \rangle \text{ or}$$
$$S = \langle 5, C+1, C+2, C+3, C+4 \rangle.$$

$$C \equiv 2 \ (mod\ 5) \Rightarrow S = \langle 5, C, C+1, C+2, C+4 \rangle.$$

$$C \equiv 3 \ (mod\ 5) \Rightarrow S = \langle 5, C, C+1, C+3, C+4 \rangle.$$

$$C \equiv 4 \ (mod\ 5) \Rightarrow S = \langle 5, C-2, C, C+2, C+4 \rangle \text{ or}$$
$$S = \langle 5, C, C+2, C+3, C+4 \rangle.$$

Arf numerical semigroups with multiplicity 6 and conductor C :

$C \equiv 0, 2, 3, 4$ or $5 \ (mod\ 6)$.

$C \equiv 0 \ (mod\ 6) \Rightarrow S = \langle 6, C+1, C+2, C+3, C+4, C+5 \rangle$ or
$S = \langle 6, 6u+2, 6u+4, C+1, C+3, C+5 \rangle$ or
$S = \langle 6, 6u+3, C+1, C+2, C+4, C+5 \rangle$ or
$S = \langle 6, 6u+4, 6u+8, C+1, C+3, C+5 \rangle$, $1 \leq u \leq (C/6)-1$.

$C \equiv 2 \ (mod\ 6) \Rightarrow S = \langle 6, 6t+2, 6t+4, C+1, C+3, C+5 \rangle$ or
$S = \langle 6, 6u+3, C, C+2, C+3, C+5 \rangle$ or
$S = \langle 6, 6u+4, 6u+8, C+1, C+3, C+5 \rangle$,
$1 \leq t \leq ((C-2)/6)$, $1 \leq u \leq ((C-2)/6)-1$.

$C \equiv 3 \ (mod\ 6) \Rightarrow S = \langle 6, 6u+3, C+1, C+2, C+4, C+5 \rangle$, $1 \leq t \leq (C-3)/6$.

$C \equiv 4 \ (mod\ 6) \Rightarrow S = \langle 6, 6t+2, 6t+4, C+1, C+3, C+5 \rangle$ or
$S = \langle 6, 6t+4, 6t+8, C+1, C+3, C+5 \rangle$, $1 \leq t \leq (C-4)/6$.

$C \equiv 5 \ (mod\ 6) \Rightarrow S = \langle 6, C, C+2, C+3, C+4, C+5 \rangle$ or
$S = \langle 6, 6t+3, C, C+2, C+3, C+5 \rangle$, $1 \leq t \leq (C-5)/6$.

$n_A(C,m)$ = the number of Arf numerical semigroups with conductor C and multiplicity m

$$n_A(C,4)=\begin{cases}\dfrac{C}{4} & ,\ C\equiv 0\ (mod\ 4)\\[2ex] \dfrac{C-2}{4} & ,\ C\equiv 2\ (mod\ 4)\\[2ex] 1 & ,\ C\equiv 3\ (mod\ 4)\end{cases}$$

$$n_A(C,5)=\begin{cases}2 & ,\ C\equiv 0\ (mod\ 5)\ \ or\ \ C\equiv 4\ (mod\ 5)\\ 1 & ,\ C\equiv 2\ (mod\ 5)\ \ or\ \ C\equiv 3\ (mod\ 5)\end{cases}$$

$$n_A(C,6)=\begin{cases}\dfrac{C}{2}\text{-}2 & ,\ C\equiv 0\ (mod\ 6)\\[2ex] \dfrac{C-2}{2} & ,\ C\equiv 2\ (mod\ 6)\\[2ex] \dfrac{C-3}{6} & ,\ C\equiv 3\ (mod\ 6)\\[2ex] \dfrac{C-4}{3} & ,\ C\equiv 4\ (mod\ 6)\\[2ex] \dfrac{C+1}{6} & ,\ C\equiv 5\ (mod\ 6)\end{cases}$$

Arf numerical semigroups with multiplicity 6 and conductor 30:

$\langle 6, 31, 32, 33, 34, 35 \rangle$

$\langle 6, 8, 10, 31, 33, 35 \rangle$

$\langle 6, 14, 16, 31, 33, 35 \rangle$

$\langle 6, 20, 22, 31, 33, 35 \rangle$

$\langle 6, 26, 28, 31, 33, 35 \rangle$

$\langle 6, 9, 31, 32, 34, 35 \rangle$

$\langle 6, 15, 31, 32, 34, 35 \rangle$

$\langle 6, 21, 31, 32, 34, 35 \rangle$

$\langle 6, 27, 31, 32, 34, 35 \rangle$

$\langle 6, 10, 14, 33, 34, 35 \rangle$

$\langle 6, 16, 20, 31, 33, 35 \rangle$

$\langle 6, 22, 26, 31, 33, 35 \rangle$

$\langle 6, 28, 31, 32, 33, 35 \rangle$

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Thank You