Huneke-Wiegand Conjecture for Numerical Semigroup Rings

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Based on a joint work in progress with

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Torsions in Tensor products

If R is a commutative Noetherian domain and M is an R-module, then

$$T(M) := \{m \in M \; ; \; rm = 0 \text{ for some non-zero } r \in R\}$$

is an R-module and is called the torsion submodule of M.

We say that M is toresion free, when $T(M) = 0$.

Question

When the tensor product of two modules over R is torsion-free?

More precisely, for which modules and which classes of rings, the assumption that $M \otimes_R N$ is torsion-free can be considered as a criterion for projectivity in either M or N?

- Auslander (1961), gave a rather complete answer when R is an unramified regular local ring.
- Lichtenbaum (1966), proved the ramified case.

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Conjecture (Huneke-Wiegand, 1994)

Let R be a one dimensional Gorenstein local domain and M be a finitely generated R-module. If M is not free, then

$$T(M \otimes_R \mathrm{Hom}_R(M, R)) \neq 0.$$

Conjecture (Auslander-Reiten, 1975)

Let R be a commutative Noetherian local ring, and let M be a finitely generated R-module. If M is not free, then $\mathrm{Ext}^i_R(M, M \oplus R) \neq 0$ for some $i > 0$.

Theorem (Celikbas-Takahashi, 2009)

If the Huneke-Wiegand Conjecture holds, then the Auslander-Reiten Conjecture holds over Gorenstein domains of any dimension.

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Let R be a Cohen-Macaulay local ring of dimension one with a canonical module K_R and let I be a faithful ideal of R.

Conjecture (Goto-Takahashi-Taniguchi-Truong, 2015)

If $I \otimes_R \mathrm{Hom}_R(I, K_R)$ is torsion free, then I is isomorphic to either R or K_R as an R-module.

They answered to their conjecture as the following :

✓ Let $\overline{R}$ be the integral closure of R in the total ring $Q(R)$ of fractions. If $\overline{R}$ is a finitely generated R-module, the conjecture holds.

✓ They have example that the result does not remain true, if we remove the finiteness condition on $\overline{R}$.

✓ Assume that $e(R) \leq 6$, where $e(R)$ is the multiplicity of R. Then the conjecture holds.

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Numerical semigroup interpretation

- $S = \{r_1 n_1 + \cdots + r_d n_d;\ r_i \geq 0\}$,
  numerical semigroup minimally generated by
  $M(S) = \{n_1 < \cdots < n_d\}$
- $R = k[t^{n_1}, \ldots, t^{n_d}]$,
  the associated numerical semigroup ring.
- $A = \{a_1, \ldots, a_n\} + S$,
  a relative ideal of S minimally generated by $n = \mu(A)$ elements of $\mathbb{N}$.
- $A^{-1} = \{s \in \mathbb{Z}\ ;\ s + a \in S \ \text{for all} \ a \in A\}$,
  is again a relative ideal of S.

$$\mu(A)\mu(A^{-1}) \geq \mu(A + A^{-1})$$

Herzinger (1996)

Let $I = t^{a_1}R + \cdots + t^{a_n}R$. If $\mu(A)\mu(A^{-1}) > \mu(A + A^{-1})$, then $T(I \otimes_R \operatorname{Hom}_R(I, R)) \neq 0$.

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- $A = \{0, y\} + S$
  the relative ideal of S generated by $\{0, y > 0\}$.
- $A_0^{-1} = A^{-1} \cup \{0\} = \{0, s \in S; s + y \in S\}$,
  considered as a numerical semigroup with minimal generating set $M(A_0^{-1})$.

$$|M(A_0^{-1})| \geq \mu(A^{-1}).$$

Theorem

Let $I = R + t^y R$. Then the length of $T(I \otimes_R \mathrm{Hom}_R(I, R))$ is equal to the number of elements $x \in M(A_0^{-1})$ such that $x + 2y \in S$.

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Herzinger (1999), Herzinger-Sanford (2004)

If the multiplicity of S is smaller than or equal to 8 and $\mu(A) \geq 2$, then $\mu(A)\mu(A^{-1}) > \mu(A + A^{-1})$.

Example

Let $S = \langle 10, 14, 15, 21 \rangle$ and $A = \{0, 1\} + S$. Then

$$S = \{ \quad 0, 10, 14, 15, 20, 21, 24, 25, 28, 29, 30, 31, 34, 35, 36, 38, \\ 39, 40, 41, 42, 43, 44, 45, 46, 48, \rightarrow\}.$$

$A^{-1} = \{14, 20\} + S$ and $A + A^{-1} = \{14, 15, 20, 21\} + S$. Hence

$$\mu(A) = \mu(A^{-1}) = 2\ ,\ \mu(A + A^{-1}) = 4.$$

$A_0^{-1} = \langle 14, 20, 24, 29, 30, 35, 39, 41, 45, 51 \rangle$ and

$$\{29, 39, 41, 51\} = \{x \in M(A_0^{-1}); x + 2 \in S\}.$$

and so the ideal $I = R + t^y R$ satisfies the conjecture.

S is called symmetric if $S = \{F(S) - s \,;\; s \in \mathbb{Z} \setminus S\}$, where $F(S) = \max(\mathbb{Z} \setminus S)$ is the Frobenius number of S.
It is well known that R is Gorenestein if and only if S is symmetric (Kunz, 1970).

Problem 1

For every (symmetric) numerical semigroup S and any integer $y > 0$ there exists an element in the minimal set of generators of $(\{0, y\} + S)_0^{-1}$ such that $x + 2y \in S$.

Solving the above problem

$\Updownarrow$

proving the Huneke-Wiegand Conjecture
for two generated monomial ideals over numerical semigroup rings.

Let $S = \langle m_1 < \cdots < m_d \rangle$, with Frobenius number $F(S)$. For an integer $y > 0$, we are looking for an element $x \in M(A_0^{-1})$ such that $x + 2y \in S$.

- ✓ If $y > F$ or $m_d > F$, then $m_d + y, m_d + 2y \in S$ and $x := m_d$.

- ✓ Let $T := S \cup F(S)$. Then $F(S) > F(T)$ is an element of the minimal generating set of T, with $F(S) + y, F(S) + 2y \in T$.

- ✓ Let $a = \min\{s \in S : s + y \in S\}$ and $b = \min\{s \in S : s + 2y \in S\}$. If $b \leq a$ (e.g. $m_1 + 2y \in S$), then

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- ✓ Let $m_1 \leq 8$, then the conjecture holds for all monomial ideals of R (Herzinger, 1999 & Herzinger-Sanford, 2004).

- ✓ If S is a complete intersection, then the conjecture holds for two generated monomial ideals of S (García Sánchez-Leamer, 2013).

- ✓ If $S = \langle a, a+1, \ldots, 2a-2 \rangle$, for $a \geq 3$, then the conjecture holds for all monomial ideals of R (Goto-Takahashi-Taniguchi-Truong, 2015).

- ✓ Let $m_1 \leq 7$ and I be a non-zero monomial ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, K_R)$ is torsionfree, then one has either $I \cong R$ or $I \cong K_R$ (Goto-Takahashi-Taniguchi-Truong, 2015).

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Lemma

Let S be a numerical semigroup and $y > 0$ be an integer. Then

$$T = \{0\} \cup \{x \in S : x + y \in S \text{ or } x - y \in S\}$$

is a numerical semigroup and $\{0, y\} + S$ is Huneke-Wiegand if and only if $\{0, y\} + T$ is Huneke-Wiegand.

Let $A = \{0, y\} + S$, then we may assume that

$$S = \langle a_1, a_1 + y, \ldots, a_n, a_n + y \rangle,$$

for some $a_1, \ldots, a_n \in A^{-1}$.

As $F(A_0^{-1}) = F(S)$, we state the following problem:

Problem 2

Let $S_1 = \langle a_1, \ldots, a_n \rangle$ be a numerical semigroup, $y > 0$ be an integer such that $a_i + y \notin S_1$ for all $i = 1, \ldots, n$ and let $S = S_1 + \langle a_1 + y, \ldots, a_n + y \rangle$. Then

$$F(S_1) \gneqq F(S).$$

Problem 2 $\Rightarrow$ Problem 1

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For every numerical semigroup S and any integer $y > 0$ there exists an element in the minimal set of generators of $(\{0, y\} + S)_0^{-1}$ such that $x + 2y \in S$.

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Irreducible numerical semigroups

A numerical semigroup is called *irreducible*, if it cannot be written as the intersection of two numerical semigroups properly containing it.

Symmetric numerical semigroups are those irreducible ones S, such that $F(S)$ is odd and $F(S) - x \in S$ for all $x \in \mathbb{Z} \setminus S$.

Blanco-Rosales, 2013

By adding and removing certain elements of an irreducible numerical semigroup S, one may get a new irreducible numerical semigroup $\overline{S}$ with the same Frobenius number $F := F(\overline{S}) = F(S)$ and larger multiplicity $m(\overline{S}) > m(S)$, provided $m(S) < \frac{F(S)}{2}$. Continuing in this way, we get the numerical semigroup

$$C(F) = \begin{cases} \{0, \frac{F(S)+1}{2}, \rightarrow\} \setminus \{F\} & \text{if } F \text{ is odd,} \\ \{0, \frac{F(S)}{2} + 1, \rightarrow\} \setminus \{F\} & \text{if } F \text{ is even.} \end{cases}$$

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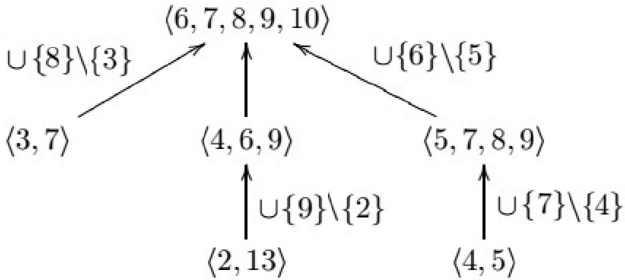
Consider the directed graph $\mathcal{G}(F) = (V, E)$ where V is the set of all irreducible numerical semigroups with Frobenius number F and $(T, S) \in E$ if $m(T) < \frac{F}{2}$ and $S = \overline{T}$.

Blanco-Rosales, 2013

$\mathcal{G}(F)$ is a tree with root $C(F)$ and the children of each vertex T are those $\overline{T}$ coming from the above procedure

$\mathcal{G}(11)$:



Theorem

Let S be an irreducible numerical semigroup with Frobenius number $F = F(S)$. If S is not a leaf of the tree $\mathcal{G}(F)$, then any two generated relative ideal of S satisfies the Huneke-Wiegand Conjecture.

Test computations by GAP

Our equivalent statement of the conjecture,

"finding a minimal generator x of $(\{0, y\} + S)_0^{-1}$ such that $x + 2y \in S$",

can be easily implemented in GAP to see that the conjecture holds for all numerical semigroup with Frobenius number at most 31.

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