Numerical Semigroup Algebra

Joint with Kee, Mee-Kyoung

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Introduction

- We are interested in properties of numerical semigroup rings, such as Cohen-Macaulyness, Gorensteiness and complete intersection.
- We emphasize morphisms rather than objects.
- Apéry sets are relative notion.
- Gluing is an operation of adding radicals.

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Numerical semigroup rings

- Let S be a numerical semigroup and κ be a field. Being a subring of the power series ring $\kappa[[\mathbf{u}]]$, the numerical semigroup ring $\kappa[[\mathbf{u}^S]]$ consists of power series $\sum_{s \in S} a_s \mathbf{u}^s$ with coefficients $a_s \in \kappa$.
- Let t be the smallest non-zero number of S. The monoid S/t is called a normalized numerical semigroup.
- A numerical semigroup is a monoid generated by finitely many positive rational numbers. If S is generated by $s_1, \cdots, s_n$, then we write $\kappa[[\mathbf{u}^S]] = \kappa[[\mathbf{u}^{s_1}, \cdots, \mathbf{u}^{s_n}]]$.
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Apéry monomials

- Given a numerical semigroup algebra R/R', a monomial is called *Apéry* if it is not divisible by any non-trivial coefficient.
- As an R'-module, R is generated by Apéry monomials.
- A monomial of R/R' written as $\mathbf{v}^s\mathbf{u}^w$, where $\mathbf{v}^s$ is a coefficient and $\mathbf{u}^w$ is an Apéry monomial, is called a *representation* of the monomial.
- A monomial may have different representations. For instance, the monomial $\mathbf{e}^{13/5}$ in $\kappa[[\mathbf{e}, \mathbf{e}^{6/5}, \mathbf{e}^{7/5}, \mathbf{e}^{8/5}]]/\kappa[[\mathbf{e}, \mathbf{e}^{6/5}]]$ has representations $\mathbf{e}\,\mathbf{e}^{8/5}$ and $\mathbf{e}^{6/5}\mathbf{e}^{7/5}$.
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Cohen-Macaulayness

- A numerical semigroup algebra is called *Cohen-Macaulay* if it is flat and the fibers are Cohen-Macaulay.
- A numerical semigroup algebra is Cohen-Macaulay if and only if it is flat.

Proposition

A numerical semigroup algebra is Cohen-Macaulay if and only if every monomial has a unique representation.

- For any $s \in S$, the algebra $\kappa[[\mathbf{u}^S]]/\kappa[[\mathbf{u}^s]]$ is Cohen-Macaulay.
- In particular, the algebra $\kappa[[\mathbf{e}^S]]/\kappa[[\mathbf{e}]]$ is Cohen-Macaulay for a normalized S.
- Let S and T be numerical semigroups generated by integers. Let $p \in T$ and $q \in S$ be relative prime numbers. Then $\kappa[[\mathbf{u}^{pS+qT}]]/\kappa[[\mathbf{u}^{qT}]]$ is Cohen-Macaulay.

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Theorem

Let R/R' and R'/R'' be numerical semigroup algebras.

- An Apéry monomial of R/R'' is the product of an Apéry monomial of R/R' and an Apéry monomial of R'/R''.
- If the algebra R/R' is flat, the product of an Apéry monomial of R/R' and an Apéry monomial of R'/R'' is an Apéry monomial of R/R''.
- If the algebra R/R' is flat, different products give rise to different Apéry monomials.

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- If the algebra R/R' is flat, different products give rise to different Apéry monomials.

Gorensteiness

- A numerical semigroup algebra is called *Gorenstein* if it is flat and the fibers are Gorenstein.
- $\kappa[[\mathbf{e}^S]]$ is a Gorenstein ring if and only if $\kappa[[\mathbf{e}^S]]/\kappa[[\mathbf{e}]]$ is a Gorenstein algebra .
- Given any $s_1, s_2 \in S$, the algebra $\kappa[[\mathbf{u}^S]]/\kappa[[\mathbf{u}^{s_1}]]$ is Gorenstein if and only if $\kappa[[\mathbf{u}^S]]/\kappa[[\mathbf{u}^{s_2}]]$ is Gorenstein.
- Among non-trivial Apéry monomials of a numerical semigroup algebra, maximal elements with respect to the order given by divisibility are called *maximal monomials*.

Proposition (Symmetry)

A flat numerical semigroup algebra is *Gorenstein* if and only if it has a unique maximal monomial.

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- $\kappa[[\mathbf{e}^S]]$ is a Gorenstein ring if and only if $\kappa[[\mathbf{e}^S]]/\kappa[[\mathbf{e}]]$ is a Gorenstein algebra .
- Given any $s_1, s_2 \in S$, the algebra $\kappa[[\mathbf{u}^S]]/\kappa[[\mathbf{u}^{s_1}]]$ is Gorenstein if and only if $\kappa[[\mathbf{u}^S]]/\kappa[[\mathbf{u}^{s_2}]]$ is Gorenstein.
- Among non-trivial Apéry monomials of a numerical semigroup algebra, maximal elements with respect to the order given by divisibility are called *maximal monomials*.

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A flat numerical semigroup algebra is *Gorenstein* if and only if it has a unique maximal monomial.

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Gorensteiness

Theorem

Let R/R' and R'/R'' be flat numerical semigroup algebras. Then R/R'' is Gorenstein if and only if R/R' and R'/R'' are Gorenstein.

Complete intersection

- A flat numerical semigroup algebra R/R' is called *complete intersection*, if there is a local surjective R'-algebra homomorphism $R'[[Y_1, \cdots, Y_n]] \to R$, whose kernel is generated by n elements.
- For a complete intersection algebra R/R', the kernel of any local surjective R'-algebra homomorphism $R'[[Z_1, \cdots, Z_m]] \to R$ is generated by m elements.
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Complete intersection

Theorem

Let R/R' and R'/R'' be flat numerical semigroup algebras. Then R/R'' is complete intersection if and only if R/R' and R'/R'' are complete intersection.

Radicals

- Let S be a numerical semigroup and $t \in \mathbb{Q}$. There exists an integer p such that $pt \in S$. We may think $\kappa[[\mathbf{u}^S, \mathbf{u}^t]]$ to be the numerical semigroup ring obtained from $\kappa[[\mathbf{u}^S]]$ by adding the p-th radical of the monomial $\mathbf{u}^{pt}$.
- Let S and T be numerical semigroups. There exists an integer p such that $pT \subset S$. We may think $\kappa[[\mathbf{u}^S, \mathbf{u}^T]]$ to be the numerical semigroup ring obtained from $\kappa[[\mathbf{u}^S]]$ by adding the p-th radicals of monomials in $\kappa[[\mathbf{u}^{pT}]]$.
- Let S and T be numerical semigroups contained in $\mathbb{N}$. Let $p \in \mathbb{N}$ and $q \in S$. The numerical semigroup ring $\kappa[[\mathbf{w}^{pS+qT}]]$ is isomorphic to $\kappa[[\mathbf{u}^{S+(qT)/p}]]$, which is obtained from $\kappa[[\mathbf{u}^S]]$ by adding the p-th radical of the monomials in $\kappa[[\mathbf{u}^{qT}]]$.

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Radicals

For $S, T \subset \mathbb{N}$ and relatively prime numbers $p \in T$ and $q \in S$, we have flat algebras:

$$\begin{array}{ccccc} & & \kappa[[\mathbf{w}^{pS+qT}]] & & \\ & \nearrow & & \nwarrow & \\ \kappa[[\mathbf{w}^{pS}]] & & & & \kappa[[\mathbf{w}^{qT}]] \\ & \nwarrow & & \nearrow & \\ & & \kappa[[\mathbf{w}^{pq}]] & & \end{array}$$

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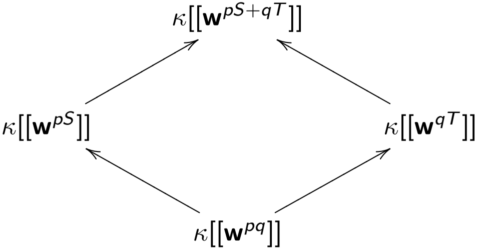

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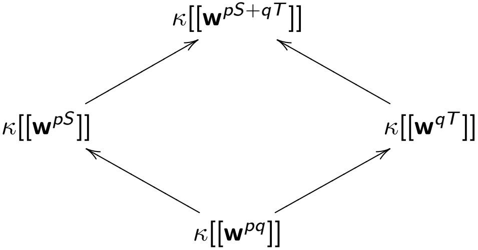

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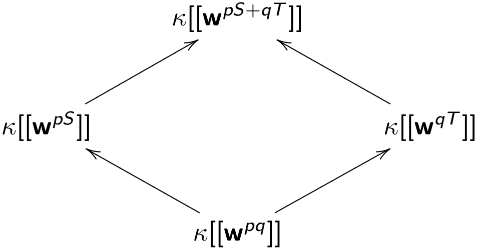

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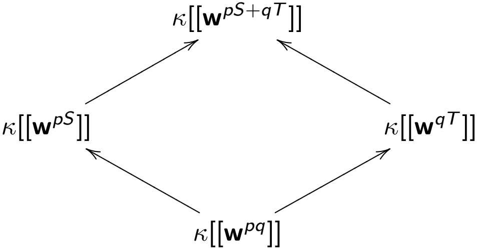

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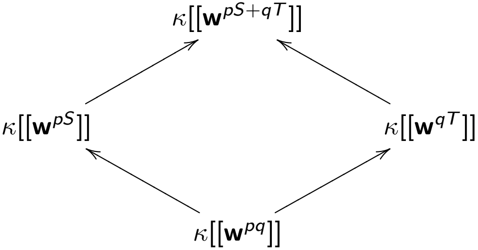

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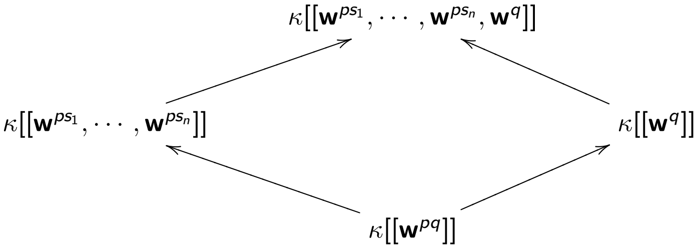

(T. Numata)

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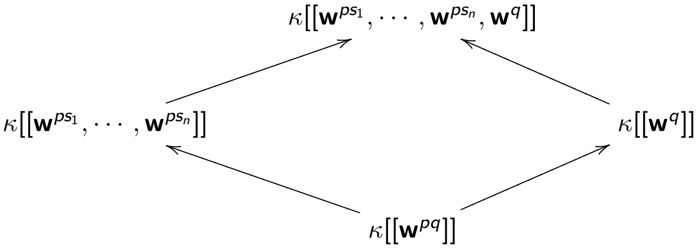

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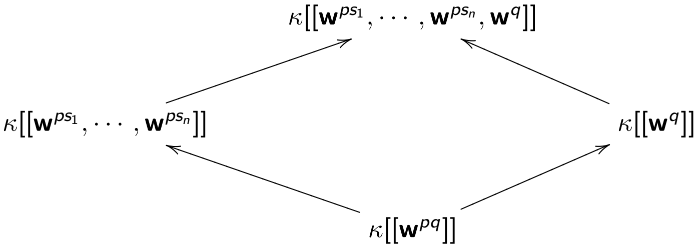

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