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Lefschetz Properties of Gorenstein Graded Algebras associated to the Apéry Set of a Numerical Semigroup

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0. Lefschetz Properties.

Let K be a field with $char(K) = 0$ and let $A = \bigoplus_{i\geq 0}^{D} A_i$ be a standard graded Artinian K-algebra with $A_0 = K$.

- A has the Weak Lefschetz Property (WLP) if $\exists L \in A_1$:

$$\times L : A_i \to A_{i+1}$$

  has maximal rank for every $i = 0, \ldots, D-1$.

- A has the Strong Lefschetz Property (SLP) if $\exists L \in A_1$:

$$\times L^d : A_i \to A_{i+d}$$

  has maximal rank for every $i = 0, \ldots, D$ and $d = 0, \ldots, D-i$.

A such linear element $L \in A_1$ for which the SLP (WLP) holds is said to be a Strong (Weak) Lefschetz Element.

Let I be an homogeneous ideal of $K[x_1, \ldots, x_n]$. We will assume the ring $A = \bigoplus_{i \geq 0}^{D} A_i \cong \dfrac{K[x_1, \ldots, x_n]}{I}$ to be Gorenstein, and hence

$$dim_K(A_i) = dim_K(A_{D-i}), \quad \forall i.$$

The integer $n = dim_K(A_1)$ is said the codimension of A.

Well known results and conjectures: (Maeno-Watanabe)

- If $n = 2$, then A has the SLP.

- If $n = 3$ and A is a CI, then A has the WLP.
- For every n, if A is a CI and I is a monomial ideal, then A has the SLP.
- It is conjectured that if A is a CI, then A has the SLP.
- If $n = 3$, it is unknown if there exist A without the WLP and the SLP.

1. Standard graded algebras associated to Apéry Sets.

Let $S = \langle g_1, g_2, \ldots, g_n \rangle \subseteq \mathbb{N}$ be a numerical semigroup.

Assuming $GCD(g_1, \ldots, g_n) = 1$, we have $|\mathbb{N} \setminus S| < \infty$
(in this case $f :=\max(\mathbb{N} \setminus S)$ is said the Frobenius number).

Then the Apéry Set of S, defined as

$\mathrm{Ap}(S) := \{s \in S : s - g_1 \notin S\} = \{0 = w_1 < w_2 < \cdots < w_m = f + g_1\}$
is a finite set and $|\mathrm{Ap}(S)| = g_1$.

Consider the homomorphism:

$$\Phi : K[x_1, \ldots, x_n] \longrightarrow K[t]$$

$$x_i \to t^{g_i}$$

The ring $R = K[S] := K[t^{g_1}, \ldots, t^{g_n}] \cong \dfrac{K[x_1, \ldots, x_n]}{ker(\Phi)}$ is called the semigroup ring associated to S.

Take $s \in S$. A representation of s is an n-uple $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $s = \sum_{i \geq 1}^{n} \lambda_i g_i$. The order of s is

$$\text{ord}(s) := max\{\sum_{i \geq 1}^{n} \lambda_i : \lambda \text{ is a representation of } s\}.$$

A representation of s is said maximal if $\text{ord}(s) = \sum_{i \geq 1}^{n} \lambda_i$.

We can associate to the representations of $s \in S$ a monomial in R by the correnspondence

$$s = \sum_{i \geq 1}^{n} \lambda_i g_i \longleftrightarrow x^{\lambda} := x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

Let $\overline{R} = R/x_1R = \langle x^\lambda \mid \sum_{i\geq 1}^n \lambda_i g_i \in \mathrm{Ap}(S)\rangle_K$. Define

$$A = \mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{R})$$

as the Associated graded algebra of the Apéry Set of S.

We have $A = \bigoplus_{i\geq 0}^D A_i = \langle x^\lambda \mid \sum_{i\geq 1}^n \lambda_i g_i \in \mathrm{Ap}(S) \text{ and } \lambda \text{ is maximal } \rangle_K$.

The artinian standard graded ring A is Gorenstein $\Longleftrightarrow$ $\mathrm{Ap}(S) = \{0 = w_1 < w_2 < \cdots < w_m = f + g_1\}$ is such that for $0 \leq i \leq m$,

$$\omega_i + \omega_{m-i} = \omega_m$$

and

$$\mathrm{ord}(\omega_i) + \mathrm{ord}(\omega_{m-i}) = \mathrm{ord}(\omega_m).$$

In this case S is said M-pure symmetric (Bryant).

Example.

$S = \langle 8, 10, 11, 12 \rangle$

$\mathrm{Ap}(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$ and S is M-pure symmetric.

Associating $10 \longleftrightarrow y$, $11 \longleftrightarrow z$, $12 \longleftrightarrow w$, we construct

$$A = K \oplus \langle y, z, w \rangle K \oplus \langle yz, yw, zw \rangle K \oplus \langle yzw \rangle K.$$

Since $22 = 11 + 11 = 10 + 12$, then in the ring A, $yw \equiv z^2$ and hence

$$A \cong \frac{K[y, z, w]}{(y^2, z^2 - yw, w^2)}.$$

2. How to compute Lefschetz Properties when A is Gorenstein. **(Maeno-Watanabe, Gondim-Zappalá)**

Let $Q := K[X_1, \ldots, X_n]$ where $X_i := \dfrac{\partial}{\partial x_i}$.

Then, there exists a polynomial $F \in K[x_1, \ldots, x_n]$ such that

$$A \cong Q/Ann_Q(F).$$

Let $1 \leq d \leq [D/2]$ and $\mathcal{B}_d = \{\alpha_i\}_{i \geq 0}^{s}$ a K-linear basis of A_d. The d-th Hessian of F is defined as the matrix

$$Hess^d(F) := \{(\alpha_i(X)\alpha_j(X)F(x))_{i,j=1}^{s}\}.$$

For $\{\alpha_i\}_{i=1}^{s_1}$ and $\{\beta_j\}_{j=1}^{s_2}$ basis of A_d and A_t define the mixed Hessians of F as

$$Hess^{d,t}(F) := \{(\alpha_i(X)\beta_j(X)F(x))\}.$$

Set $k := [D/2]$ where D is the socle degree of the algebra A.

- $A = Q/Ann_Q(F)$ has the SLP $\iff$ $Hess^d(F)$ have maximal rank for every $1 \leq d \leq [D/2]$.

- If D is an odd number, $A = Q/Ann_Q(F)$ has the WLP $\iff$ $Hess^k(F)$ has maximal rank.

- If D is an even number, $A = Q/Ann_Q(F)$ has the WLP $\iff$ $Hess^{k-1,k}(F)$ has maximal rank.

When A is associated to the Apéry Set of a Numerical Semigroup $S = \langle g_1, g_2, \ldots, g_n \rangle$, we have the following results:

- The polynomial F is $F = \sum_{\lambda \in \Lambda} x^{\lambda}$ where Λ is the set of the maximal representations of the maximal element of $\mathrm{Ap}(S)$, $f + g_1$.

- Call $\omega_1 < \omega_2 < \ldots < \omega_b$ the elements of $\mathrm{Ap}(S)$ of order d and $H := Hess^d(F)$. The entry of the matrix

  $H_{ij} \neq 0 \iff \omega_i + \omega_j \in \mathrm{Ap}(S)$ and $\mathrm{ord}(\omega_i + \omega_j) = \mathrm{ord}(\omega_i) + \mathrm{ord}(\omega_j)$.

- If $H_{ij} \neq 0$, then $H_{ij} = \sum_{\lambda \in \Lambda} x^{\lambda - \mu}$ where μ is a maximal representation of $\omega_i + \omega_j$.

$\forall \omega \in \mathrm{Ap}(S)$, its symmetric element is

$$\omega' := \omega_m - \omega = f + g_1 - \omega.$$

• If D is odd , $H := Hess^k(F)$ and $\omega_1 < \omega_2 < \ldots < \omega_b$ are the elements of $\mathrm{Ap}(S)$ of order k, then

$$H_{ij} = x_l \quad \text{if} \quad \omega_i + g_l = \omega'_j$$

otherwise $H_{ij} = 0$.

• If D is even , $H := Hess^{k-1,k}(F)$, $\omega_1 < \omega_2 < \ldots < \omega_b$ are the elements of $\mathrm{Ap}(S)$ of order $k-1$ and $v_1 < v_2 < \ldots < v_b$ are the elements of $\mathrm{Ap}(S)$ of order k, then

$$H_{ij} = x_l \quad \text{if} \quad \omega_i + g_l = v'_j$$

otherwise $H_{ij} = 0$.

Examples.

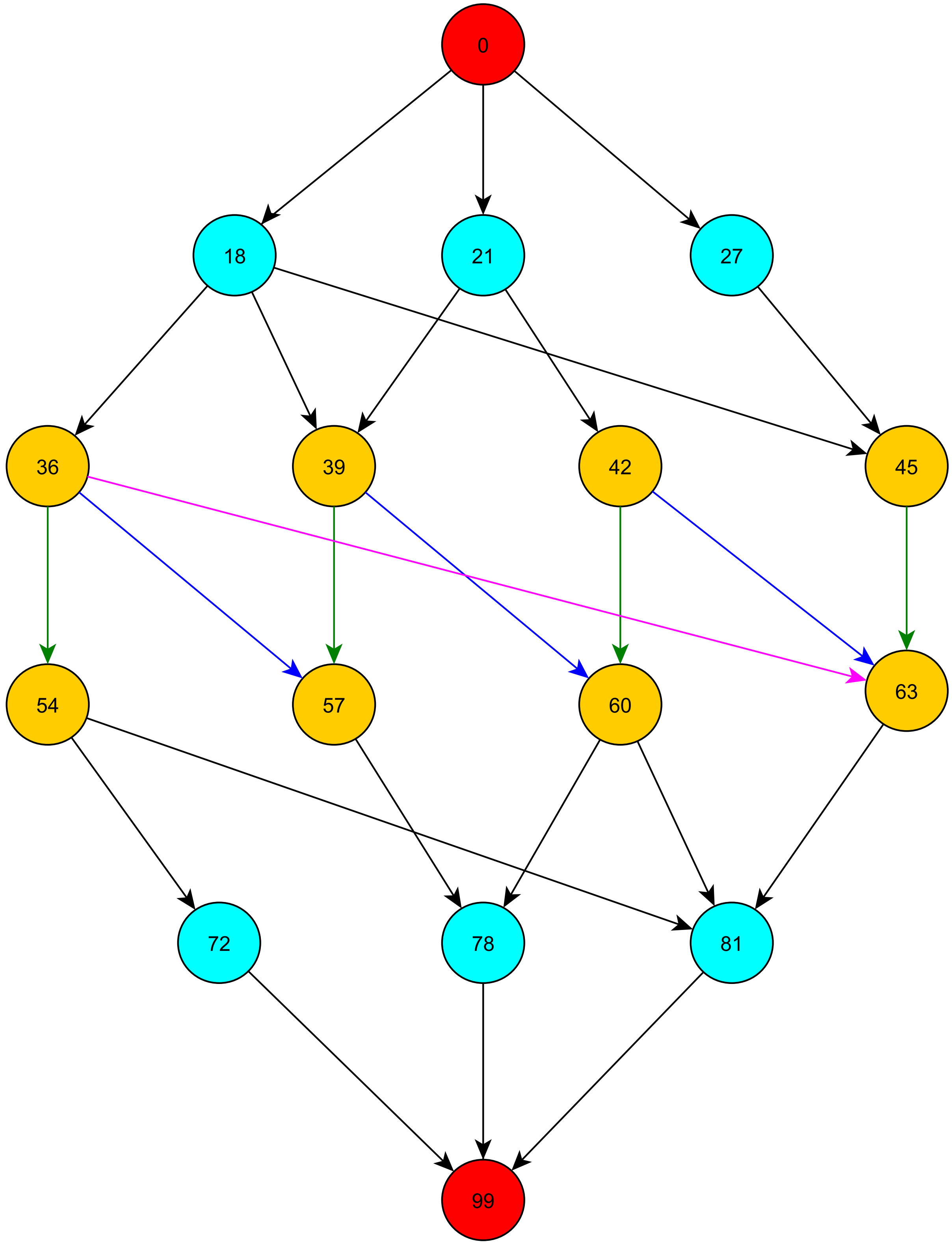
- $S = \langle 16, 18, 21, 27 \rangle$

$$A \cong \frac{K[y,z,w]}{(y^5, z^3 - y^2w, w^2, zw, y^3z)}.$$

$D = 5$ and $F = y^4w + y^2z^3$. We compute the first Hessian

$$Hess^1(F) = \begin{pmatrix} y^2w + z^3 & yz^2 & y^3 \\ yz^2 & zy^2 & 0 \\ y^3 & 0 & 0 \end{pmatrix}$$

For the second and maximal Hessian we look at the links in Ap(S) in the following graph:


0
18
21
27
36
39
42
45
54
57
60
63
72
78
81
99

The second Hessian is

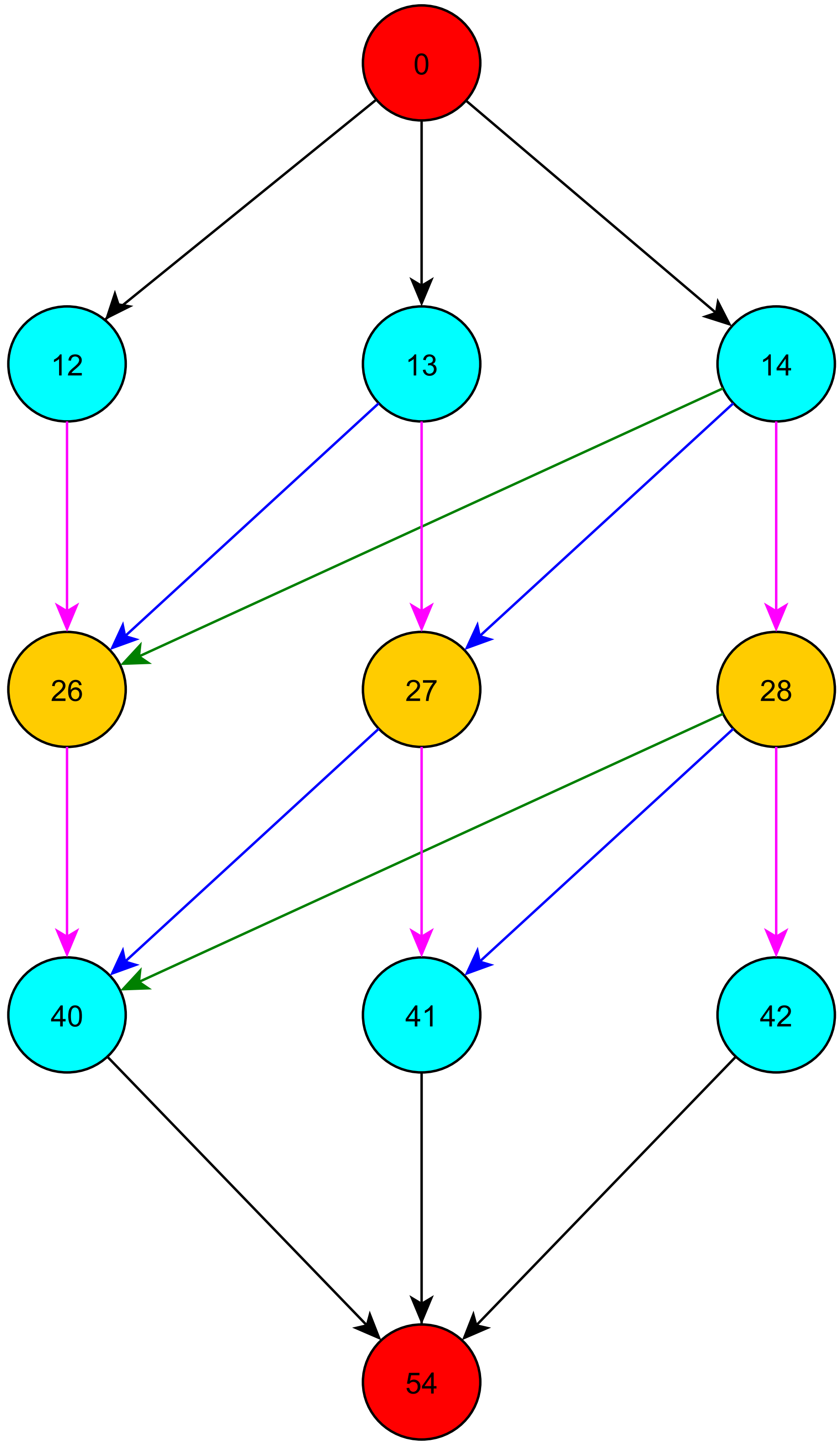
$$Hess^2(F) = \begin{pmatrix} w & 0 & z & y \\ 0 & z & y & 0 \\ z & y & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix}$$

The hessians have both maximal rank, hence A has the SLP.

- $S = \langle 11, 12, 13, 14 \rangle$

$$A \cong \frac{K[y,z,w]}{(y^2, z^2 - yw, w^4, yz, zw^3)}.$$

$D = 4$ and $F = yw^3 + z^2w^2$. We look at the links in Ap(S) in the following graph in order to compute the mixed Hessian:


0
12
13
14
26
27
28
40
41
42
54

The mixed Hessian is

$$Hess^{1,2}(F) = \begin{pmatrix} 0 & 0 & w \\ 0 & w & z \\ w & z & y \end{pmatrix}$$

and it has maximal rank, hence A has the WLP.

3. The Complete Intersection Case.

Let $S = \langle g_1, g_2, \ldots, g_n \rangle$ be a numerical semigroup such that the ring $A \cong \dfrac{K[x_2, \ldots, x_n]}{I}$ associated to Ap(S) is Gorenstein.

For $2 \leq i \leq n$, define:

$\beta_i :=$ max$\{h \in \mathbb{N} \,|\, hg_i \in$ Ap(S) and ord$(hg_i) = h\}$;
$\gamma_i :=$ max$\{h \in \mathbb{N} \,|\, hg_i \in$ Ap(S), ord$(hg_i) = h$ and
hg_i has a unique maximal representation$\}$.

$\forall\ i = 2, \ldots, n$, $\gamma_i \leq \beta_i$ and always $\gamma_2 = \beta_2$ and $\gamma_n = \beta_n$.

We define also two hyper-rectangles in $\mathbb{N}^{n-1}$:

$$B = \left\{ \sum_{i=2}^{n} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \beta_i \right\}$$

$$\Gamma = \left\{ \sum_{i=2}^{n} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \gamma_i \right\}$$

D'Anna, Micale, Sammartano proved that:

- $\mathrm{Ap}(S) \subseteq \Gamma \subseteq B$.
- A is CI $\iff$ $\mathrm{Ap}(S) = \Gamma$.
- A is CI and its defining ideal I is monomial $\iff$ $\mathrm{Ap}(S) = B$.

- The defining ideal I of A always contains the ideal

$$\tilde{I} = (x_i^{\gamma_i+1} - \rho_i \prod_{j\neq i} x_j^{\lambda_j} : \ i = 2\ldots,n)$$

where $\beta_i = \gamma_i \ \Rightarrow \ \rho_i = 0$ and $\beta_i > \gamma_i \ \Rightarrow \ \rho_i = 1$
(in this case $(\gamma_i + 1)g_i = \sum_{j\neq i} \lambda_j g_j$ are two different maximal representation of the same element) and

$$I = \tilde{I} \Longleftrightarrow A \text{ is CI.}$$

Examples.

- $S = \langle 15, 21, 35 \rangle$

$\mathrm{Ap}(S) = \{0, 21, 35, 42, 56, 70, 63, 77, 91, 84, 98, 112, 119, 133, 154\}$

$84 = 21 \cdot 4 \in \mathrm{Ap}(S)$ and $105 = 21 \cdot 5 \notin \mathrm{Ap}(S)$, $70 = 35 \cdot 2 \in \mathrm{Ap}(S)$ and $105 = 35 \cdot 3 \notin \mathrm{Ap}(S)$, hence $\beta_2 = \gamma_2 = 4$, $\beta_3 = \gamma_3 = 2$ and we can verify that

$$B = \left\{ \sum_{i=2}^{3} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \beta_i \right\} = \mathrm{Ap}(S).$$

The associated graded algebra is

$$A = \frac{K[y, z]}{(y^5, z^3)}.$$

- $S = \langle 8, 10, 11, 12 \rangle$ as in the example in the last section.

$\mathsf{Ap}(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$

$20 = 10 \cdot 2 \notin \mathsf{Ap}(S)$, $24 = 12 \cdot 2 \notin \mathsf{Ap}(S)$, $33 = 11 \cdot 3 \in \mathsf{Ap}(S)$, $44 = 11 \cdot 4 \notin \mathsf{Ap}(S)$ and $11 \cdot 2 = 10 + 12$. Hence $\beta_2 = \gamma_2 = 1$, $\beta_4 = \gamma_4 = 1$ but $1 = \gamma_3 < \beta_3 = 3$.

$$\Gamma = \left\{ \sum_{i=2}^{4} \lambda_i g_i \,|\, 0 \leq \lambda_i \leq \gamma_i \right\} = \mathsf{Ap}(S) \subsetneq B.$$

The associated graded algebra is

$$A = \frac{K[y, z, w]}{(y^2, z^2 - yw, w^2)}.$$

Proposition. Let $A = \bigoplus_{d \geq 0}^{D} A_d$ be Complete Intersection and $k = [D/2]$. Then,

$$\exists\ \gamma_i > k \Rightarrow A \text{ has the WLP.}$$

Proof (Idea). If $\gamma_i > k$, then $\forall\ \omega \in \mathrm{Ap}(S)$ of order k, $\omega + g_i \in \mathrm{Ap}(S)$ and has order $k+1$.

Hence a square submatrix of maximal dimension of $Hess^k(F)$ (or of $Hess^{k-1,k}(F)$) has x_i on all the entries on the secondary diagonal and therefore A has the WLP.

4. Not Complete Intersection Algebras in codimension 3.

Let $n = 4$ and let $S = \langle g_1, g_2, g_3, g_4 \rangle$ be an M-pure symmetric numerical semigroup. Assume the algebra A associated to Ap(S) to be Gorenstein but not CI.

Hence Ap(S) $\subsetneq \Gamma$ and

$$(\gamma_3 + 1)g_3 = \mu_2 g_2 + \mu_4 g_4$$

with $1 \leq \mu_2 \leq \gamma_2$ and $1 \leq \mu_4 \leq \gamma_4$ and $\mu_2 + \mu_4 = \gamma_3 + 1$.

A is a quotient of the Complete Intersection ring

$$G = \bigoplus_{i \geq 0}^{D} G_i = \frac{K[y, z, w]}{(y^{\gamma_2+1}, z^{\gamma_3+1} - y^{\mu_2} w^{\mu_4}, w^{\gamma_4+1})}.$$

The following result characterize the ring $A = \frac{K[x_2, \dots, x_n]}{J} = \bigoplus_{d \geq 0}^{D-C} A_d$ in function of his socle degree $D - C$.

Theorem.

- $1 \leq C \leq \gamma_3$.
- $\Gamma \setminus \mathrm{Ap}(S) = \{\omega \in \Gamma \text{ s.t. } \omega + C g_3 \notin \Gamma\}$.
- Set $h_2 = \gamma_2 - \mu_2 + 1$, $h_3 = \gamma_3 - C + 1$ and $h_4 = \gamma_4 - \mu_4 + 1$. The defining ideal of A is

$$J = I + (z^{h_3} y^{h_2}, z^{h_3} w^{h_4}),$$

where I is the defining ideal of G.

Moreover $A \cong \dfrac{G}{(0 :_G z^C)}$.

The proof is based on the fact that A is Gorenstein and hence Ap(S) must be symmetric.

Example.

Starting from

$$G = \frac{K[y,z,w]}{(y^3, z^3 - y^2w, w^2)},$$

we can construct

$$A_1 = \frac{G}{(0 :_G z)} = \frac{K[y,z,w]}{(y^3, z^3 - y^2w, w^2, z^2y, z^2w)}$$

and

$$A_2 = \frac{G}{(0 :_G z^2)} = \frac{K[y, z, w]}{(y^3, z^3 - y^2w, w^2, zy, zw)}.$$

The following are the generators of these algebras as K-vector spaces, in blue the generators that are nonzero in all the three rings, in green the generators nonzero in G and A_1, in red the generators nonzero only in G.

$1;\quad y, z, w;\quad y^2, yz, yw, z^2, zw;\quad y^2z, y^2w \cong z^3, yz^2, yzw, z^2w;$
$y^2z^2, y^2zw, yz^2w;\quad y^2z^2w.$

5. WLP in codimension 3.

$G = \dfrac{K[y,z,w]}{(y^{\gamma_2+1}, z^{\gamma_3+1} - y^{\mu_2}w^{\mu_4}, w^{\gamma_4+1})}$ has the WLP (CI in codimension 3). We show that

$$A = \frac{G}{(0 :_G z^C)}$$

has the WLP for $1 \leq C \leq \gamma_3$. Since

$$\frac{G}{(0 :_G z^C)} = \frac{\dfrac{G}{(0 :_G z^{C-1})}}{(0 : z)}$$

it suffices to show:

Theorem. $A = \dfrac{G}{J} = \dfrac{G}{(0 :_G z)}$ has the WLP .

Proof (Idea).

Let $L = a_2y + a_3z + a_4w \in G_1 = A_1$ be a Weak Lefschetz Element for G and let $k := [D/2]$. Without loss of generality we can assume $a_2, a_3, a_4 \neq 0$.

(1) D odd: The map $\times L : G_k \to G_{k+1}$ is an isomorphism and since $G \cong A \oplus J$ as K-vector spaces, it is easy to show that $\times L : A_k \to A_{k+1}$ is surjective.

(2) D even: The map $\times L : G_{k-1} \to G_k$ is injective and we want to prove that $\times L : A_{k-1} \to A_k$ is also injective. Hence we want

$$L(A_{k-1}) \cap J = (0).$$

Since $J = (0 :_G z)$, we need to show $zL(f) \neq 0$ for all $f \in A_{k-1}$.

The matrix associated to $\times L : G_{k-1} \to G_k$ is ${}^tH(a_2y, a_3z, a_4w)$ where $H := Hess_{k-1,k}(F)$.

The matrix associated to $\times z : G_k \to G_{k+1}$ is $H(0, z, 0)$. Thus the square matrix

$$Z = H(0, z, 0)\,{}^tH(a_2y, a_3z, a_4w)$$

is associated to $\times zL : G_{k-1} \to G_{k+1}$.

$G_{k-1} \cong A_{k-1} \oplus J_{k-1}$. Using some linear algebra we show that $Ker(Z) = J_{k-1}$. This implies $zL(f) \neq 0$ for all $f \in A_{k-1}$.

Thanks for the attention!