Puiseux Monoids and Their Atomic Structure

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Outline

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1. **Basic Notions**
2. **Atomicity Conditions**
3. **Bounded Puiseux Monoids**
4. Monotone Puiseux Monoids

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What is a Puiseux monoid?

Definition

A Puiseux monoid is an additive submonoid of $\mathbb{Q}_{\geq 0}$.

Remark: Puiseux monoids are a generalization of numerical semigroups. However, the former are not necessarily

- finitely generated;
- atomic.

Example: For a prime p, consider the Puiseux monoid

$$M = \langle 1/p^n \mid n \in \mathbb{N} \rangle.$$

The set of atoms of M is empty, i.e., $\mathcal{A}(M) = \emptyset$; hence M is not atomic. In addition, M fails to be finitely generated.

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Intuition from Numerical Semigroups

Every numerical semigroup is finitely generated, while:

Observation (1)

A Puiseux monoid is finitely generated iff it is isomorphic to a numerical semigroup.

Numerical semigroups are atomic and minimally generated, while:

Observation (2)

A Puiseux monoid is atomic iff it is minimally generated.

Numerical semigroups have a unique minimal generating set, while:

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If a Puiseux monoid has a minimal generating set, then such a generating must be unique.

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Examples

Let P denote the set of primes.

Example 1: The Puiseux monoid $M = \langle 1/p \mid p \in P \rangle$ is atomic, and $\mathcal{A}(M) = \{1/p \mid p \in P\}$. Therefore $|\mathcal{A}(M)| = \infty$.

Example 2: Let M be the Puiseux monoid generated by the set $S \cup T$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and $T = \{1/p \mid n \in P \setminus \{2\}\}$. It follows that M is not atomic; however, $\mathcal{A}(M)$ is the infinite set T.

Example 3 If $\{d_n\}$ is a sequence of natural numbers such that $d_n \mid d_{n+1}$ properly for every $n \in \mathbb{N}$, then $M = \langle 1/d_n \mid n \in \mathbb{N} \rangle$ is a Puiseux monoid satisfying $\mathcal{A}(M) = \emptyset$; this is because

$$\frac{1}{d_n} = \frac{d_{n+1}}{d_n} \frac{1}{d_{n+1}} \quad \text{for every} \quad n \in \mathbb{N}.$$

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Sufficient Conditions for Atomicity

- For $r \in \mathbb{Q}\setminus\{0\}$, we denote by $\mathsf{n}(r)$ (resp., $\mathsf{d}(r)$) the positive numerator (reps., denominator) when r is represented as a reduced fraction.
- For $R \subseteq \mathbb{Q}\setminus\{0\}$, we define the *numerator set* (resp., *denominator set*) of R to be $\mathsf{n}(R) = \{\mathsf{n}(r) \mid r \in R\}$ (resp., $\mathsf{d}(R) = \{\mathsf{d}(r) \mid r \in R\}$).

Proposition (1)

Let M be a Puiseux monoid. Then $\mathsf{d}(M\setminus\{0\})$ is bounded iff M is atomic (indeed, isomorphic to a numerical semigroup).

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Let M be a Puiseux monoid. If 0 is not a limit point of M, then M is atomic.

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Existence of Nontrivial Atomic Submonoids

As we have seen before, not every Puiseux monoid is atomic. However, every Puiseux monoid contains a nontrivial atomic submonoid.

Theorem

If M is Puiseux monoid, then it satisfies exactly one of the following conditions:

- *M is isomorphic to a numerical semigroup;*
- *M contains an atomic submonoid with infinitely many atoms.*

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Realizability of $|\mathcal{A}(M)|$

Theorem

For every $m \in \mathbb{N}_0 \cup \{\infty\}$, there exists a Puiseux monoid M such that $|\mathcal{A}(M)| = m$.

Sketch of Proof:

- For $m = 0$, we can take $M = \langle 1/p^n \mid n \in \mathbb{N} \rangle$, where p is a prime.
- Let $m \in \mathbb{N}$. For distinct primes p and q, define

$$M = \left\langle m, \dots, 2m-1, \frac{q}{p^{m+1}}, \frac{q}{p^{m+2}}, \dots \right\rangle.$$

  If $q > m$, then $\mathcal{A}(M) = \{m, \dots, 2m-1\}$ and so $|\mathcal{A}(M)| = m$.
- Finally, suppose $m = \infty$. Let P denote the set of primes, and take $M = \langle 1/p \mid p \in P \rangle$. Then $\mathcal{A}(M) = \{1/p \mid p \in P\}$ and so $|\mathcal{A}(M)| = \infty$. $\square$

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Bounded Puiseux Monoids

Definition

Let M be a Puiseux monoid.

- We say that M is *bounded* if it can be generated by a bounded subset of rationals.
- We say that M is *strongly bounded* if it can be generated by a subset of rationals R such that $\mathrm{n}(R)$ is bounded.

Observations:

1. Every strongly bounded Puiseux monoid is bounded.
2. If P denotes the set of primes, then $M = \langle \frac{p-1}{p} \mid p \in P \rangle$ is bounded but not strongly bounded.
3. If P denotes the set of primes, then $M = \langle \frac{p^2-1}{p} \mid p \in P \rangle$ is not bounded.

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Antimatter Puiseux Monoids

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A Puiseux monoid M is said to be *antimatter* if $\mathcal{A}(M)$ is empty.

Recall: If $\{d_n\} \subset \mathbb{N}$ such that $d_n \mid d_{n+1}$ properly, then $M = \langle 1/d_n \mid n \in \mathbb{N} \rangle$ satisfies that $\mathcal{A}(M) = \emptyset$, i.e., M is antimatter. The next result is a generalization of this fact.

Definition: The *spectrum* of a sequence $\{a_n\}$ is the set of primes p such that $p \mid a_n$ for every n large enough.

Theorem

Let $\{r_n \mid n \in \mathbb{N}\}$ *be a strongly bounded subset of rationals generating* M. *If* $\mathrm{d}(r_n)$ *divides* $\mathrm{d}(r_{n+1})$, *the sequence* $\{\mathrm{d}(r_n)\}$ *is unbounded, and the spectrum of* $\{\mathrm{n}(r_n)\}$ *is empty, then* M *is antimatter.*

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Finite Puiseux Monoid

Definition

A Puiseux monoid M is said to be finite if there are only finitely many primes dividing elements of $\mathrm{d}(M)$.

Example: If P denotes the set of primes and $p \in P$, then $\langle 1/p^n \mid n \in \mathbb{N} \rangle$ is finite, but $\langle 1/q \mid q \in P \rangle$ is not.

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Let M be a strongly bounded finite Puiseux monoid. Then M is atomic iff M is isomorphic to a numerical semigroup.

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Monotone Puiseux Monoid

We say that a subset of $\mathbb{R}$ is increasing (resp., decreasing) if we can list its elements increasingly (resp., decreasingly).

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A Puiseux monoid M is said to be *increasing* (resp., *decreasing*) if it can be generated by an increasing (resp., decreasing) set of rationals. A Puiseux monoid is *monotone* if it is either increasing or decreasing.

Observations:

- Increasing Puiseux monoids are atomic.
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Prime Reciprocal Puiseux Monoid

Definition

A Puiseux monoid M is *prime reciprocal* if there exists a subset of primes P such that $M = \langle 1/p \mid p \in P \rangle$.

Theorem (G-Gotti)

Every submonoid of a reciprocal Puiseux monoid is atomic.

Remark: In particular, a prime reciprocal Puiseux monoid is atomic. The next question suggests itself.

Question: Are the submonoids of an atomic Puiseux monoid atomic?

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Definition

For $r \in \mathbb{Q}_{>0}$, we call *multiplicative r-cyclic* to the Puiseux monoid generated by the positive powers of r, and we denote it by M_r, that is $M_r = \langle r^n \mid n \in \mathbb{N} \rangle$.

The next theorem describes the atomic structure of multiplicatively cyclic Puiseux monoids.

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For $r \in \mathbb{Q}_{>0}$, let M_r be the multiplicative r-cyclic Puiseux monoid. Then the following statements hold.

- *If $\mathrm{d}(r) = 1$, then M_r is atomic with $\mathcal{A}(M_r) = \{\mathrm{n}(r)\}$.*
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End of Presentation

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THANK YOU FOR YOUR KIND ATTENTION!