Asymptotic ω-primality in commutative monoids

J. I. García García

Universidad de Cádiz

July 6, 2016

Ongoing work with D. Marín Aragón and A. Vigneron-Tenorio

Motivations

- Defined and studied for integral domains ([1])
- There exists a method to compute ω-primality in cancellative monoids ([3])
- Periodicity of ω-primality has been studied in [6]
- Asymptotic ω-primality for numerical monoids is studied in [4]

Definitions

- S is a f. g. commutative cancellative monoid without units
- $S \cong \mathbb{N}^p / \sim_M$ with M a subgroup of $\mathbb{Z}^p$
- If S is a numerical semigroup, M has $rank(M) = p - 1$ and M is defined by an homogeneous equation $a_1 x_1 + \cdots + a_p x_p = 0$
- For every $s \in S$ there exists $\gamma \in \mathbb{N}^p$ such that $[\gamma]_{\sim_M} = [\gamma] = s$
- Set of factorizations of $s = [\gamma]$ and its multiples:

$$\mathsf{Z}(s) = \{x \in \mathbb{N}^p \mid x - \gamma \in M\} = (\gamma + M) \cap \mathbb{N}^p$$
$$\mathsf{Z}(s + S) = ((\gamma + M) + \mathbb{N}^p) \cap \mathbb{N}^p$$

- $S \cong \mathbb{N}^p / \sim_M$ is atomic and $\mathcal{A}(S) = \{[e_1], \ldots, [e_p]\}$
- Define the functions

$$|\cdot| : \mathbb{Q}^p \to \mathbb{Q},\ |(x_1, \ldots, x_p)| = x_1 + \cdots + x_p (\text{lenght function}),$$
$$\Pi : \mathbb{Q}^p \to \mathbb{Q}^p_{\geq},\ \Pi(\textstyle\sum_{i=1}^p \lambda_i e_i) = \sum_{\lambda_i > 0} \lambda_i e_i$$

- $\Gamma^n = \gamma + \frac{1}{n} M$, $\frac{1}{n} M = \{\frac{1}{n} m \mid m \in M\}$, $\frac{1}{n}\mathbb{N}^p = \{\frac{1}{n} x \mid x \in \mathbb{N}^p\}$, $\frac{1}{n}\mathbb{Z}^p = \{\frac{1}{n} x \mid x \in \mathbb{Z}^p\}$

ω-primality

Definition (See Definition 1.1 in [2])

Let S be an atomic monoid with set of units $S^\times$ and set of irreducibles $\mathcal{A}(S)$. For $s \in S \setminus S^\times$, we define $\omega(s) = m$ if m is the smallest positive integer with the property that whenever $s | a_1 + \cdots + a_t$, where each $a_i \in \mathcal{A}(S)$, there is a $T \subseteq \{1, 2, \ldots, t\}$ with $|T| \leq n$ such that $s | \sum_{k \in T} a_k$. If no such m exists, then $\omega(s) = \infty$. For $x \in S^\times$, we define $\omega(x) = 0$.

Proposition (Proposition 3.3 in [3])

Let $S \cong \mathbb{N}^p / \sim_M$ and $s = [\gamma] \in S$. Then

$$\omega(s) = \max\{|x| \mid x \in \mathrm{Minimals}_{\leq} \mathsf{Z}(s + S)\}$$

$\omega(s)$ is the length of the farthest element of $\mathrm{Minimals}_{\leq} \mathsf{Z}(s + S)$ to the origin with the norm $|x|_1 = \sum |x_i|$

Properties of ω

Among the properties of ω we have (see [1] and [5]):

1. $\omega([\gamma]) \leq \omega([2\gamma]) \leq 2\omega([\gamma])$
2. $\omega([\gamma] + [\gamma']) \leq \omega([\gamma]) + \omega([\gamma'])$ (subadditive)
3. $\omega((n+m)[\gamma]) \leq \omega(n[\gamma]) + \omega(m[\gamma])$. The sequence $\{\omega(n[\gamma])\}_{n\in\mathbb{N}}$ is subadditive and by [7]:

   $$\lim_{n\to\infty} \omega(n[\gamma])/n = \inf\{\omega(n[\gamma])/n\}$$

   This is the asymptotic ω-primality of s (or $\overline{\omega}(s)$)
4. if $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ verifies that $f_{n-1}|f_n$, then

   $$\omega(f_n[\gamma])/f_n = \omega(kf_{n-1}[\gamma])/kf_{n-1} \leq \omega(f_{n-1}[\gamma])/f_{n-1}$$

   For instance, $\{\omega(2^n[\gamma])/2^n\}_{n\in\mathbb{N}}$ is a decreasing sequence and $\lim\{\omega(2^n[\gamma])/2^n\} = \lim\{\omega(n[\gamma]/n)\} = \overline{\omega}(s)$

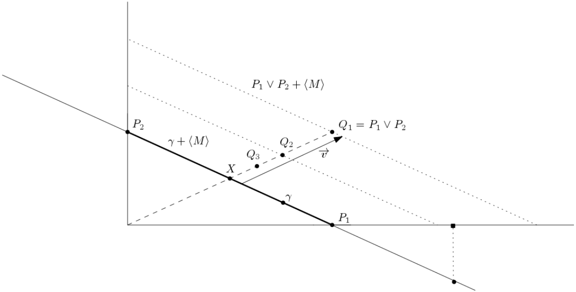
Asymptotic ω-primality

We have $\omega(n[\gamma])/n = \max\{|x| \ : \ x \in \mathrm{Minimals}_{\leq} \Pi(\Gamma^n)\}$, and, therefore,

$$\overline{\omega}([\gamma]) = \lim_{n\to\infty} \max\{|x| \ : \ x \in \mathrm{Minimals}_{\leq} \Pi(\Gamma^n)\}$$

(it is not necessary to add $\frac{1}{n}\mathbb{N}^p$)
($\Gamma^n = \gamma + \frac{1}{n}M$)

In [4], $\overline{\omega}$-primality is computed for numerical monoids



Let $S = \langle a_1 < a_2 \rangle$, $\gamma = (\gamma_1, \gamma_2)$, $s = \gamma_1 s_1 + \gamma_2 s_2$, $\langle M \rangle$ is the vect. subspace generated by M

The elements of $\mathrm{Minimals}_{\leq} \Pi(\Gamma^n)$ are between the hyperplanes $\gamma + \langle M \rangle$ and $Q_i + \langle M \rangle$ with $Q_i = X + \frac{1}{n}\vec{v}$

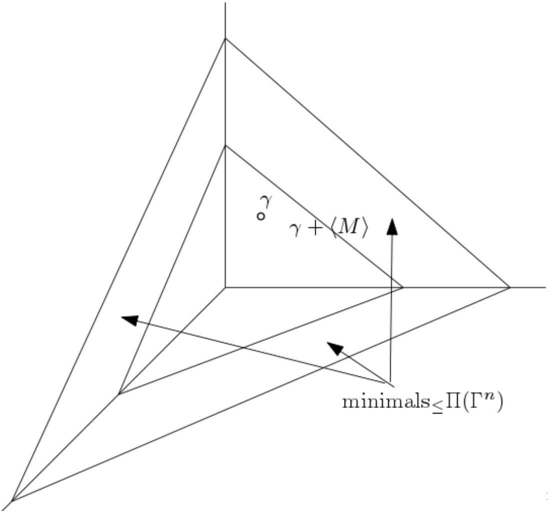
$$\overline{\omega}(s) = \overline{\omega}([\gamma]) = \max\{|P_1|, |P_2|\} = |P_1| = s/a_1$$

with $P_1 = (s/a_1, 0)$, $P_2 = (0, s/a_2)$

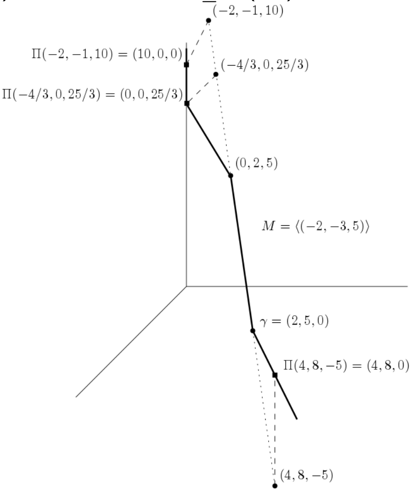
For a numerical semigroup

$$\overline{\omega}(s) = s/a_1$$

and $P_1 = (s/a_1, 0, \ldots, 0)$ verifies $\overline{\omega}(s) = |P_1|$
Since we only use $\gamma + \langle M \rangle$, the method is valid for every group M with $\mathrm{rank}\,(M) = p - 1$ (congruences can be removed)



If $S \cong \mathbb{N}^p / \sim_M$ with $\mathrm{rank}\,(M) = 1$, $\mathrm{Minimals}_{\leq} \Pi(\Gamma^n)$ are in the



polygonal $\Pi(\gamma + \langle M \rangle) \subset \mathbb{Q}^p_{\geq}$
$(0,2,5) \in \mathbb{Z}([2,5,0])$, $\omega([2,5,0]) = 10$

- if $n = 1$, $\Gamma^1 = (2,5,0) + M = \{\dots, (4,8,-5), (2,5,0), (0,2,5), (-2,-1,10), \dots\}$
- if $n = 2$, $\Gamma^2 = (2,5,0) + \frac{1}{2}M = \{(4,8,-5), (3,6.5,-2.5), (2,5,0), (1,3.5,2.5), (0,2,5), (-1,0.5,7.5), (-2,-1,10), \dots\}$
- Taking the map $\sigma : \mathbb{N} \to \mathbb{N}$, $\sigma(n) = 3^n$ we obtain that $(0,0,25/3), (0,2,5), (2,5,0) \in \Pi(\Gamma^{\sigma(n)})$

Thus, $\mathrm{Minimals}_{\leq} \Pi(\Gamma^{3^n}) \subset \overline{(0,0,25/3)(0,2,5)} \cup \overline{(2,5,0)(2,5,0)}\}$.
Thus, $\overline{\omega}([2,5,0]) = 25/3$

Ideas for computing $\overline{\omega}([\gamma])$:

- Make a partition with 2^p elements of $\mathbb{Q}^p$ attending the signal of its elements
- For each of these subsets compute the set of minimals $\Pi(\gamma + \langle M \rangle)$

For every subset Δ of $\{1, \ldots, p\}$ define the set

$$\mathfrak{Q}_\Delta = \{x \in \mathbb{Q}^p \mid x_i \geq 0 \ \forall i \in \Delta, x_i \leq 0 \ \forall i \in \{1, \ldots, p\} \setminus \Delta\}.$$

We use the following notation: for every $\Delta \subset \{1, \ldots, p\}$, intersect Γ^n and $\mathfrak{Q}_\Delta$ and denote $\Gamma^n \cap \mathfrak{Q}_\Delta$ by Γ^n_Δ

Note that the set $\gamma + \langle M \rangle$ is an affine variety (affine subspace) of $\mathbb{Q}^p$, and $\langle M \rangle$ is defined by the set of homogeneous equations of M. Take into account the following considerations:

1. For every Δ the set $\mathfrak{Q}_\Delta$ is a cone.
2. The sets $\gamma + \langle M \rangle$ and $\Gamma_\Delta = (\gamma + \langle M \rangle) \cap \mathfrak{Q}_\Delta \subset \mathbb{Q}^p$ are polyhedrons (intersection of a set of half-spaces), and the projection $\Pi(\Gamma_\Delta)$ is also a polyhedron.
3. For every $n \in \mathbb{N}$, $\Gamma_\Delta^n \subset \Gamma_\Delta$ and $\Pi(\Gamma_\Delta^n) \subset \Pi(\Gamma_\Delta)$.
4. In general, $\mathrm{minimals}_{\leq} \Pi(\Gamma_\Delta^n)$ is not a subset of $\mathrm{minimals}_{\leq} \Pi(\Gamma_\Delta)$, but

$$\lim \max\{|x| \ : \ x \in \mathrm{Minimals}_{\leq} \Pi(\Gamma_\Delta^n)\} = \max\{|x| \ : \ x \in \mathrm{Minimals}_{\leq} \mathrm{Vertices}(\Pi(\Gamma_\Delta))\} \quad (1)$$

Computing the asymptotic ω-primality of an element.
`Input:` A system of generators of M and $\gamma \in \mathbb{N}^p$.
`Output:` $\overline{\omega}([\gamma])$.

1. For every $\Delta \subset \{1, \ldots, p\}$ compute $\mathrm{Vertices}(\Pi(\Gamma_\Delta))$
2. Compute $W = \cup_{\Delta \subset \{1,\ldots,p\}} \mathrm{Minimals}_{\leq} \mathrm{Vertices}(\Pi(\Gamma_\Delta))$ and $V = \mathrm{minimals}_{\leq} W$
3. Return $\overline{\omega}([\gamma]_{\sim_M}) = \max\{|v| : v \in V\}$

If there exists x_0 such that $|x_0| = \max\{|v| : v \in V\}$ and $x_0 \in W$, then the algorithm returns $\overline{\omega}([\gamma])$ (this condition is always satisfied when $\mathrm{rank}\,(M) = 1$ and $\mathrm{rank}\,(M) = p - 1$)

$\gamma = (1,1,1,1),\ M = \langle (3,0,1,-4), (-9,1,0,8) \rangle,$

$$M \equiv \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - 7x_2 + 5x_3 + 2x_4 = 0 \end{cases}$$

$$W = \left\{ \left(\frac{29}{8}, \frac{3}{8}, 0, 0\right), (7,0,0,0), \left(0,0,0,\frac{19}{3}\right), \left(\frac{19}{4}, 0,0,0\right), \left(0, \frac{7}{9}, 0, \frac{29}{9}\right), \left(0, \frac{19}{12}, \frac{29}{12}, 0\right) \right\} \quad (2)$$

Their lenghts are: $4, 7, 19/3, 19/4, 4, 4,$
but $(7,0,0,0)$ is not minimal

$$V = \left\{ \left(\frac{29}{8}, \frac{3}{8}, 0, 0\right), \left(0,0,0,\frac{19}{3}\right), \left(\frac{19}{4}, 0,0,0\right), \left(0, \frac{7}{9}, 0, \frac{29}{9}\right), \left(0, \frac{19}{12}, \frac{29}{12}, 0\right) \right\}, \quad (3)$$

their lenghts are $\{4, 19/3, 19/4, 4, 4\}$, and the first three elements of $\{\omega(2^n[(1,1,1,1)])\}$ are $\{7, 7, 6.5\}$. Hence,
$\overline{\omega}([(1,1,1,1)]) = 19/3$

Thank you!

D. F. Anderson and S. T. Chapman.
How far is an element from being prime?
J. Algebra Appl., 9(5):779–789, 2010.

D. F. Anderson, S. T. Chapman, N. Kaplan, and D. Torkornoo.
An algorithm to compute ω-primality in a numerical monoid.
Semigroup Forum, 82(1):96–108, 2011.

V. Blanco, P. A. García-Sánchez, and A. Geroldinger.
Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids.
Illinois J. Math., 55(4):1385–1414 (2013), 2011.

J. I. García-García, M. A. Moreno-Frías, and A. Vigneron-Tenorio.
Computation of the ω-primality and asymptotic ω-primality with applications to numerical semigroups.
Israel J. Math., 206(1):395–411, 2015.

A. Geroldinger and W. Hassler.
Local tameness of v-Noetherian monoids.
J. Pure Appl. Algebra, 212(6):1509–1524, 2008.

C. O'Neill and R. Pelayo.
On the linearity of ω-primality in numerical monoids.
2013.

G. Szegö.
Bemerkungen zu einer Arbeit von Herrn M. Fekete: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten.
Math. Z., 21(1):203–208, 1924.