The Milnor number of plane irreducible singularities in positive characteristic

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In this talk we present some results of

- E. García Barroso and A. Płoski, *The Milnor number of plane irreducible singularities in positive characteristic*, Bull. London Math. Soc. 48 (2016) 94-98.

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For any power series $f, h \in \mathbf{K}[[x, y]]$ we define the **intersection multiplicity** $i_0(f, h)$ by putting

$$i_0(f, h) = \dim_{\mathbf{K}} \mathbf{K}[[x, y]]/(f, h),$$

where (f, h) is the ideal of $\mathbf{K}[[x, y]]$ generated by f and h.

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Property

Let f, h be non-zero power series without constant term. Then $i_0(f, h) < +\infty$ if and only if $\{f = 0\}$ and $\{h = 0\}$ have no common branch.

First definitions: semigroup of a branch

Properties

- $i_0(f, h_1 h_2) = i_0(f, h_1) + i_0(f, h_2)$.
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For any irreducible power series $f \in \mathbf{K}[[x, y]]$, where $\mathbf{K}$ is an algebraically closed field of characteristic $p \geq 0$, we put

$$\Gamma(f) = \{i_0(f, h) \ : \ h \text{ runs over all power series such that } h \not\equiv 0 \pmod{f}\}.$$

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$$\Gamma(f) = \{i_0(f, h) \ : \ h \text{ runs over all power series such that } h \not\equiv 0 \pmod{f}\}.$$

$\Gamma(f)$ is a semigroup called the **semigroup associated with the branch** $\{f = 0\}$.

Properties of the semigroup

Lemma

- $\Gamma(f)$ *is a numerical semigroup (i.e.* $\gcd(\Gamma(f)) = 1$*).*
- *There exists a unique sequence* $v_0, \ldots, v_g$ *such that*
  - $v_0 = \min(\Gamma(f) \setminus \{0\}) = \operatorname{ord} f$,
  - $v_k = \min(\Gamma(f) \setminus \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1})$ *for* $k \in \{1, \ldots, g\}$,
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Definition

$\Gamma(f)$ is a **tame semigroup** if p does not divide v_k for all $k \in \{0, 1, \ldots, g\}$.

Properties of the semigroup

Let $\mathbf{e_k} := \gcd(v_0, \ldots, v_k)$ for $k \in \{1, \ldots, g\}$. Then

- $e_0 > e_1 > \cdots e_{g-1} > e_g = 1$ and
- $e_{k-1} v_k < e_k v_{k+1}$ for $k \in \{1, \ldots, g-1\}$.

Let $\mathbf{n_k} := e_{k-1}/e_k$ for $k \in \{1, \ldots, g\}$. Then

- $n_k > 1$ for $k \in \{1, \ldots, g\}$ and
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Properties

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- $\Gamma(f)$ *has* **conductor**

$$c(f) = \sum_{k=1}^{g} (n_k - 1) v_k - v_0 + 1.$$

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In characteristic zero we have

$$\mu(f) = c(f),$$

for any irreducible power series $f \in \mathbf{K}[[x, y]]$, and consequently $\mu(f)$ **is determined by** $\Gamma(f)$**.**

But in positive characteristic is not, in general, true:

Example (Boubakri-Greuel-Markwig)

$f = x^p + y^{p-1}$ and $g = (1 + x)f$, where $p > 2$. Then $\Gamma(f) = \Gamma(g)$, $c(f) = c(g) = (p-1)(p-2)$ but $\mu(f) = +\infty$ and $\mu(g) = p(p-2)$.

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In positive characteristic it is well-known that $\mu(f) \geq c(f)$.

We give necessary and sufficient conditions for the equality $\mu(f) = c(f)$ in terms of the semigroup associated with f, provided that $p > v_0 = \mathrm{ord}\, f$=multiplicity of $\Gamma(f)$.

Main result

Theorem (GB-P, May 2015)

Let $f \in \mathbf{K}[[x,y]]$ *be an irreducible singularity and let* $v_0, \ldots, v_g$ *be the minimal system of generators of* $\Gamma(f)$*. Suppose that* $p = \operatorname{char} \mathbf{K} > v_0$*. Then the following two conditions are equivalent:*

- $\mu(f) = c(f)$
- $\Gamma(f)$ *is a tame semigroup (*$v_k \not\equiv 0 \pmod{p}$ *for* $k = 1, \ldots, g$*).*

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- $\mu(f) = c(f)$
- *$\Gamma(f)$ is a tame semigroup ($v_k \not\equiv 0 \pmod p$ for $k = 1, \ldots, g$).*

Example

Let $f(x, y) = (y^2 + x^3)^2 + x^5 y$. Then f is irreducible and $\Gamma(f) = 4\mathbf{N} + 6\mathbf{N} + 13\mathbf{N}$, so the conductor is $c(f) = 16$. Let $p = \operatorname{char} \mathbf{K} > v_0 = 4$. If $p \neq 13$ then $\mu(f) = c(f)$ by Theorem. If $p = 13$ then a direct calculation shows that $\mu(f) = 17$.

Ingredients of the proof

Let $f \in \mathbf{K}[[x, y]]$ be an irreducible singularity with

$$\Gamma(f) = \mathbf{N}v_0 + \cdots + \mathbf{N}v_g.$$

Since f is unitangent $i_0(f, x) = \mathrm{ord}\, f = v_0$ or $i_0(f, y) = \mathrm{ord}\, f = v_0$.

We assume that $i_0(f, x) = \mathrm{ord}\, f = v_0$.

Ingredients of the proof

We need a sharpened version of Merle's factorization theorem on polar curves:

Theorem (Factorization of the polar curve)

Suppose that $v_0 = \operatorname{ord} f \not\equiv 0 \pmod p$. *Then* $\frac{\partial f}{\partial y} = \psi_1 \cdots \psi_g$ *in* $\mathbf{K}[[x,y]]$, *where*

(i) $\operatorname{ord} \psi_k = \frac{v_0}{e_k} - \frac{v_0}{e_{k-1}}$ *for* $k \in \{1, \ldots, g\}$.

(ii) *If* $\phi \in \mathbf{K}[[x,y]]$ *is an irreducible factor of* ψ_k, $k \in \{1, \ldots, g\}$, *then*

$$\frac{i_0(f,\phi)}{\operatorname{ord} \phi} = \frac{e_{k-1} v_k}{v_0},$$

and

(iii) $\operatorname{ord} \phi \equiv 0 \left(\mathit{mod}\ \frac{v_0}{e_{k-1}} \right)$.

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Lemma

Suppose that $v_0 = \operatorname{ord} f \not\equiv 0 \pmod{p}$. *Then*

$$i_0\left(f, \frac{\partial f}{\partial y}\right) = c(f) + \operatorname{ord} f - 1.$$

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Lemma

Suppose that $p > \operatorname{ord} f$*. Then* $i_0\left(f, \frac{\partial f}{\partial y}\right) \leq \mu(f) + \operatorname{ord} f - 1$ *with equality if and only if* $v_k \not\equiv 0 \pmod p$ *for* $k \in \{1, \ldots, g\}$*.*

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Proof of main theorem: it is a consequence of Lemmas.

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What happens if we do not suppose $p = \operatorname{char}\mathbf{K} > v_0 = \operatorname{ord} f$?

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Proposición (Case $g = 1$)

If $\Gamma(f) = \mathbf{N}v_0 + \mathbf{N}v_1$ (so $c(f) = (v_0 - 1)(v_1 - 1)$) then

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The other implication is still open.