Primary decomposition of powers of prime ideals for numerical semigroups

Ralf Fröberg

July 10, 2016

General commutative algebra

An ideal P in a ring R is prime if $xy \in P$ implies that x or y is in P. Equivalently, P is prime if and only if R/P is a domain ($\bar{x} \cdot \bar{y} = \bar{0}$ implies $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$). An ideal Q is primary if $xy \in Q$, $x \notin Q$ implies $y^n \in Q$ for some $n > 0$. Equivalently, every zerodivisor in R/Q is nilpotent ($\bar{x} \cdot \bar{y} = \bar{0}$, $\bar{x} \neq 0$ implies $(\bar{y})^n = \bar{0}$ for some $n > 0$). If Q is primary, then the radical $\sqrt{Q} = \{x; x^n \in Q \text{ for some } n > 0\}$ is a prime P, one says that Q is P-primary.

If R is Noetherian (such as a polynomial ring $k[x_1, \ldots, x_n]$ or a power series ring $k[[x_1, \ldots, x_n]]$ over a field k), then every ideal I is an irredundant intersection of primary ideals, $I = Q_1 \cap \cdots \cap Q_s$, Q_i P_i-primary, where the P_i's are different and unique. If P is a maximal ideal, then P is prime and P^n are P-primary for all $n > 0$. The primary ideals belonging to minimal primes in $\{P_i\}$ are unique. If I is a graded ideal (generated by homogeneous elements) in $k[x_1, \ldots, x_n]$, then the primary ideal belonging to minimal primes are graded, and the remaining (embedded) can be chosen to be graded. If P is a prime ideal, it is no longer true that P^n must be P-primary, P^n may have embedded components.

A bit more special commutative algebra

If I is an ideal in a Noetherian ring R, then the subring $R(I) = R[It, I^2t^2, I^3t^3, \ldots]$ of $R[t]$ is called the Rees ring of I. This was introduced by Rees, who showed that $R(I)$ is Noetherian (so $R(I) = R[It, I^2t^2, \ldots, I^nt^n]$ for some n), in his proof of the Artin-Rees lemma. If P is a prime ideal, then the primary decomposition of P^n always contains a P-primary component, it is $P^{(n)} = P^nR_P \cap R$, and it is called the symbolic nth power of P. It is easy to see that $P^n \subseteq P^{(n)}$.

Cowsik asked if the symbolic Rees algebra $R_s(P) = R[Pt, P^{(2)}t^2, P^{(3)}t^3, \ldots]$ always is Noetherian. This was shown not to be true by Roberts. There are even counterexamples when $R = k[t^{n_1}, t^{n_2}, t^{n_3}] = k[x, y, z]/P$. Goto-Nishida-Watanabe showed that for $n \geq 4$ then $k[t^{7n-3}, t^{(5n-2)n}, t^{8n-3}]$ does not have a finitely generated symbolic Rees algebra if $\mathrm{char}(k) = 0$. The smallest counterexample is $k[t^{25}, t^{72}, t^{29}] = k[x, y, z]/(x^{11} - yz^7, y^3 - x^4z^4, z^{11} - x^7y^2)$.

Numerical semigroup rings

If $R = k[t^{n_1}, \ldots, t^{n_s}]$, we map $k[x_1, \ldots, x_s]$ into $k[t]$, by $x_i \mapsto t^{n_i}$. Then $R \simeq k[x_1, \ldots, x_s]/P$, and P is a prime ideal since R is a domain. In the case of numerical semigroup rings, R is 1-dimensional, so $P^{(n)} = P^n$ or $P^{(n)} \cap Q$, where Q is $(x_1, \ldots, x_s)$-primary.

Hochster has shown that if $k[x_1, \ldots, x_s]/P$ is a complete intersection, then $k[x_1, \ldots, x_s]/P^n$ is a Cohen-Macaulay ring. Thus, if the semigroup ring is a complete intersection, then $P^{(n)} = P^n$, since a Cohen-Macaulay ring has no embedded components. Thus, in this case $R_s(P) = R(P)$ is Noetherian and P^n is the primary decomposition of P^n. Huneke has shown that if $P^{(n)} = P^n$ if $n >> 0$, then P is a complete intersection.

3-generated numerical semigroups

In the sequel we mean numerical semigroup when we write semigroup. If the semigroup is generated by 3 elements, and is not a complete intersection, then $R = k[t^{n_1}, t^{n_2}, t^{n_3}] \simeq k[x, y, z]/P$ where P is generated by the three 2×2-subdeterminants of a matrix (the relation matrix)

$$\begin{pmatrix} x^{a_1} & y^{b_1} & z^{c_1} \\ z^{c_2} & x^{a_2} & y^{b_2} \end{pmatrix}.$$

Herzog and Ulrich has shown that $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $a_1 = a_2, b_1 \leq b_2, c_1 \geq c_2$ (or a permutation). Huneke has shown that if P is a 2-dimensional prime in a 3-dimensional ring, then $P^{(2)}/P^2$ is generated by one element Δ.

Schenzel has, in the case of a 3-generated semigroup, determined Δ. The result depends on whether $R_s(P) = R[Pt, P^{(2)}t^2]$ or not. If $R_s(P) = R[Pt, P^{(2)}t^2]$, then if $a_1 \leq a_2, b_1 \geq b_2, c_1 \geq c_2$.

$$\Delta = \begin{vmatrix} x^{a_1} & y^{b_1} & z^{c_1} \\ z^{c_2} & x^{a_2} & y^{b_2} \\ y^{b_1} & x^{a_2-a_1}y^{b_1-b_2}z^{c_2} & y^{a_1}z^{c_1-c_2} \end{vmatrix}.$$

He also showed that $(x, y, z)\Delta \in P^2$.

In the other case, $a_1 > a_2, b_1 > b_2, c_1 > c_2$, there is a similar result:

$$\Delta = \begin{vmatrix} x^{a_1} & y^{b_1} & z^{c_1} \\ z^{c_2} & x^{a_2} & y^{b_2} \\ x^{a_1 - a_2} & y^{b_1 - b_2} z^{c_1} & x^{a_1} z^{c_1 - c_2} \end{vmatrix}$$

and $(x, y, z)\Delta \in P^2$.

Theorem

Suppose that $R = k[t^a, t^b, t^c] = k[x, y, z]/P$ *is not a complete intersection. Then* $P^2 = ((\Delta) + P^2) \cap ((z) + P^2)$ *is a primary decomposition. If furthermore* $R_s(P) = R[Pt, P^{(2)}t^2]$, *then* $P^{2n} = (P^{(2)})^n \cap ((z^n) + P^{2n})$ *and* $P^{2n+1} = P(P^{(2)})^n \cap ((z^n) + P^{2n})$.

Proof Since $P^{(2)} = (\Delta) + P^2$ and since $(z) + P^2$ is (x, y, z)-primary, it suffices to note that $(z) \cap (\Delta) \subseteq P^2$ to see the first statement. If $R_s(P) = R[Pt, P^{(2)}t^2]$, then $P^{(2n)} = \sum_{i=0}^{n} (P^{(2)})^i P^{2n-2i} = (P^{(2)})^n$ since $P^2 \subseteq P^{(2)}$. In the same way we see that $P^{(2n+1)} = PP^{(2n)}$. Finally $(z^n) \cap P^{(2n)} = (z^n) \cap ((\Delta) + P^2)^n \subseteq P^{2n}$ since $z\Delta \subseteq P^2$.

The remaining part is a search for examples when $R_s(P) = R[Pt, P^{(2)}t^2]$.

Arithmetic sequences

Now suppose that the semigroup is generated by $m, m+d, m+2d$, $\gcd(m, m+d, m+2d) = 1$. The semigroup is symmetric (so the semigroup ring is a complete intersection) if m is even and d odd. Otherwise the relation matrix is

$$\begin{pmatrix} x^{k+d} & y & z \\ z^k & x & y \end{pmatrix}.$$

Thus $R_s(P) = R[Pt, P^{(2)}t^2]$.

Semigroups generated by $a < b < c$, $c - a \leq 4$

If the semigroup is not generated by an arithmetic sequence, the generators are $m, m+1, m+3$ or $m, m+1, m+4$ or $m, m+2, m+3$ or $m, m+3, m+4$, If the semigroup is generated by $m, m+1, m+3$, it is symmetric if $m \equiv 0 \pmod 3$. If $m = 3k+1$ the relation matrix is

$$\begin{pmatrix} x^k & y & z \\ z^k & x^2 & y^2 \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$ for all k.

If $m = 3k + 2$, $k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^2 & z \\ z^k & x^2 & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$.
For $m = 5$ the relation matrix is

$$\begin{pmatrix} x^2 & y^2 & z \\ z & x^3 & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$.

If the semigroup is generated by $m, m+1, m+4$, it is symmetric if $m \equiv 0 \pmod 4$ (and if $m = 5$).
If $m = 4k+1$, $k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^{k-1} & y & z \\ z^k & x^3 & y^3 \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$.

If $m = 4k + 2$, $k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^k & y^2 & z \\ z^k & x^3 & y^2 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ for all $k \geq 3$.

If $m = 4k + 3$ the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^3 & z \\ z^k & x^3 & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ only if $k = 1$ or $k = 2$.

If the semigroup is generated by $m, m+2, m+3$, it is symmetric if $m \equiv 0 \pmod 3$ (and if $m = 4$).

If $m = 3k + 1$ the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^2 & z^2 \\ z^{k-1} & x & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$.

If $m = 3k + 2$ the relation matrix is

$$\begin{pmatrix} x^k & y^2 & z^2 \\ z^k & x & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$ for all k.

If the semigroup is generated by $m, m+3, m+4$, it is symmetric if $m \equiv 0 \pmod 4$ (and if $m = 6$ or $m = 9$).

If $m = 4k + 1$, $k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^3 & z^3 \\ z^{k-1} & x & y \end{pmatrix}$$

and $R_s(P) \neq R[Pt, P^{(2)}t^2]$.

If $m = 4k + 2$, $k \geq 2$, the relation matrix is

$$\begin{pmatrix} x^{k+1} & y^2 & z^3 \\ z^{k-1} & x & y^2 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $k \geq 4$.

If $m = 4k + 3$ the relation matrix is

$$\begin{pmatrix} x^{k+1} & y & z^3 \\ z^k & x & y^3 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ only if $k = 3$.

Theorem

If the semigroup is generated by $a < b < c$*,* $c - a \leq 4$*, not symmetric, and* a, b, c *not an arithmetic sequence, then* $R_s(P) = R[Pt, P^{(2)}t^2]$ *if and only if the generators are 5,6,8 or 15,18,19 or 7,10,11 or 11,14,15 or* $4k+2, 4k+3, 4k+6$*,* $k \geq 3$*, or*
$4k+2, 4k+5, 4k+6$*,* $k \geq 4$*.*

Semigroups of multiplicity 3

Suppose that the semigroup is generated by $3, 3k+1, 3l+2$. In order to have a 3-generated semigroup we must have $l \leq 2k$ and $k \leq 2l+1$. The semigroup is never symmetric. The relation matrix is

$$\begin{pmatrix} x^{2l-k+1} & y & z \\ z & x^{2k-l} & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$.

Semigroups of multiplicity 4

If a 3-generated semigroup has multiplicity 4 and is not symmeteric, it has generators $4, 4k+1, 4l+3$. If $k > l$ the relation matrix is

$$\begin{pmatrix} x^{3l-k+2} & y & z^2 \\ z & x^{2k-2l-1} & y \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $5l - 3k + 3 \leq 0$. If $k \leq l$ the relation matrix is

$$\begin{pmatrix} x^{2l-2k+1} & y & z \\ z & x^{3k-l} & y^2 \end{pmatrix}$$

and $R_s(P) = R[Pt, P^{(2)}t^2]$ if and only if $3l - 5k + 1 \geq 0$.

Theorem

If the semigroup is 3-generated and has multiplicity 3, then $R_s(P) = R[Pt, P^{(2)}t^2]$*. If the multiplicity is 4 and not symmetric, it is generated by* $4, 4k+1, 4l+3$*, and* $R_s(P) = R[Pt, P^{(2)}t^2]$ *if and only if* $k > l$ *and* $5l - 3k + 3 \leq 0$ *or if* $k \leq l$ *and* $3l - 5k + 1 \geq 0$*.*