On Numerical Semigroups Generated by Squares, Cubes and Quartics of Three Consecutive Integers

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Frobenius numbers of numerical semigroups generated by three consecutive squares or cubes

M. Lepilov, J. O'Rourke, I. Swanson, (LO'RS), **Semigroup Forum, 2015**

The Frobenius numbers were calculated by Euclidean algorithm with negative and positive remainders

Theorem 3.1 *The Frobenius number of the numerical semigroup generated by 3 consecutive squares* $(4\,m)^2, (4\,m+1)^2, (4\,m+2)^2$ *is*

$F < (4\,m)^2, (4\,m+1)^2, (4\,m+2)^2 > \; = 128\,m^3 - 20\,m - 5,$ *if* $m \geq 4,$

$F < (4\,m+1)^2, (4\,m+2)^2, (4\,m+3)^2 > \; = 160\,m^3 + 128\,m^2 + 10\,m - 9,$
if $m \geq 2,$

$F < (4\,m+2)^2, (4\,m+3)^2, (4\,m+4)^2 > \; = 128\,m^3 + 224\,m^2 + 124\,m + 19,$
if $m \geq 1,$

$F < (4\,m+3)^2, (4\,m+4)^2, (4\,m+5)^2 > \; = 272\,m^3 + 648\,m^2 + 481\,m + 103,$
if $m \geq 1.$

Theorem 4.1 *The Frobenius number of the numerical semigroup generated by 3 consecutive cubes* $(18\,m+r)^3, (18\,m+r+1)^3,$ *and* $(18\,m+r+2)^3$, *reads*

$F(18m+r) = f_r(m),$ *where* $r = 0, 1, \ldots, 17$, *and* m *are :*

(1) $m \geq 0$ *if* $r = 11, 12, 13, 14, 15, 16.$
(2) $m \geq 1$ *if* $r = 0, 1, 2, 3, 4, 5, 6, 9, 10.$
(3) $m \geq 2$ *if* $r = 7.$
(4) $m \geq 4$ *if* $r = 8.$
(5) $m \geq 15$ *if* $r = 17$

$f_0(m) = 1259712\,m^5 + 320760\,m^4 + 44712\,m^3 + 4644\,m^2 + 154\,m - 1,$
$f_1(m) = 629856\,m^5 + 571536\,m^4 + 190512\,m^3 + 31752\,m^2 + 2548\,m + 75,$
$f_2(m) = 1259712\,m^5 + 1020600\,m^4 + 332424\,m^3 + 55404\,m^2 + 4698\,m + 157,$
$f_3(m) = 629856\,m^5 + 921456\,m^4 + 532656\,m^3 + 153144\,m^2 + 21812\,m + 1223,$
$f_4(m) = 1259712\,m^5 + 1720440\,m^4 + 946728\,m^3 + 263412\,m^2 + 37082\,m + 2107,$
$f_5(m) = 629856\,m^5 + 1271376\,m^4 + 968112\,m^3 + 355752\,m^2 + 63828\,m + 4499,$
$f_6(m) = 1259712\,m^5 + 2420280\,m^4 + 1840968\,m^3 + 693468\,m^2 + 129322\,m + 9537,$
$f_7(m) = 629856\,m^5 + 1621296\,m^4 + 1590192\,m^3 + 753624\,m^2 + 173908\,m + 15695,$
$f_8(m) = 1259712\,m^5 + 3120120\,m^4 + 3061800\,m^3 + 1488132\,m^2 + 358074\,m + 34087,$
$f_9(m) = 629856\,m^5 + 1971216\,m^4 + 2398896\,m^3 + 1429704\,m^2 + 419252\,m + 48539,$
$f_{10}(m) = 1259712\,m^5 + 3819960\,m^4 + 4609224\,m^3 + 2766636\,m^2 + 826058\,m + 98125,$

f_{11} *(m)* = 629856 $\mathbf{m^5}$ +2321136 $\mathbf{m^4}$ +3394224 $\mathbf{m^3}$ +2466936 $\mathbf{m^2}$ +892404 $\mathbf{m}$+128663,
f_{12} *(m)* =1259712 $\mathbf{m^5}$ +4519800$\mathbf{m^4}$ +6483240$\mathbf{m^3}$+4648212 $\mathbf{m^2}$ +1665946 $\mathbf{m}$+238803,
f_{13} (m) =629856 $\mathbf{m^5}$ +2671056 $\mathbf{m^4}$ +4482864 $\mathbf{m^3}$+3730536 $\mathbf{m^2}$ +1541908 $\mathbf{m}$+253539,
f_{14} (m) =1259712 $\mathbf{m^5}$ +5219640$\mathbf{m^4}$ +8637192$\mathbf{m^3}$ +7135452 $\mathbf{m^2}$ +2943162 $\mathbf{m}$+484897,
f_{15} (m) =629856 $\mathbf{m^5}$ +3020976 $\mathbf{m^4}$ +5758128 $\mathbf{m^3}$+5458968 $\mathbf{m^2}$ +2576660 $\mathbf{m}$+484767,
f_{16} (m)=1259712$\mathbf{m^5}$ +5919480$\mathbf{m^4}$ +11117736$\mathbf{m^3}$+10433124$\mathbf{m^2}$ +4892186$\mathbf{m}$+917039,
f_{17} (m) =629856 $\mathbf{m^5}$ +3370896 $\mathbf{m^4}$ +7126704 $\mathbf{m^3}$+7455240 $\mathbf{m^2}$ +3864564 $\mathbf{m}$+794987.

This theorem does not include all possible numerical semigroups generated by 3 consecutive cubes. The missing cases are routine to verify:

F($2^3, 3^3, 4^3$)=181, F($3^3, 4^3, 5^3$)=1098, F($4^3, 5^3, 6^3$)=2107,

F($5^3, 6^3, 7^3$)=5249, F($6^3, 7^3, 8^3$)= 10745, F($7^3, 8^3, 9^3$)= 21700,

F($8^3, 9^3, 10^3$)= 38919, F($9^3, 10^3, 11^3$)= 55222, F($10^3, 11^3, 12^3$)= 103589,

F($17^3, 18^3, 19^3$)= 881440, F($25^3, 26^3, 27^3$)= 4868957,

F($26^3, 27^3, 28^3$)= 9413533, F($35^3, 36^3, 37^3$)= 23887437,

F($44^3, 45^3, 46^3$)= 121672187, F($53^3, 54^3, 55^3$)= 171468734,

F($62^3, 63^3, 64^3$)= 656175201, F($71^3, 72^3, 73^3$)= 702420331,

F($89^3, 90^3, 91^3$)= 2107464204, F($107^3, 108^3, 109^3$)= 5184832025,

F($125^3, 126^3, 127^3$)= 11115847882, F($143^3, 144^3, 145^3$)= 21540510999,

F($161^3, 162^3, 163^3$)= 38633078456, F($179^3, 180^3, 181^3$)= 65177647909,

F($197^3, 198^3, 199^3$)= 104643740310, F($215^3, 216^3, 217^3$)= 161261882627,

F($233^3, 234^3, 235^3$)= 240099190564, F($251^3, 252^3, 253^3$)= 347134951281,

F($269^3, 270^3, 271^3$)= 489336206114,

Corollary
$F(\mathbf{n^2}, (\mathbf{n+1})^2, (\mathbf{n+2})^2) = 0(\mathbf{n^3})$, $F(\mathbf{n^3}, (\mathbf{n+1})^3, (\mathbf{n+2})^3) = 0(\mathbf{n^5})$

Remark *Some considerations lead to believe in the case of three consecutive quartics* $\boldsymbol{n^4, (n+1)^4, (n+2)^4}$ *we would require 88 formulas, whereas experimental tests make us believe that we need 40 formulas.*

The goal of the work: *for numerical semigroups*

$$R_n^k = \langle (n-1)^k, n^k, (n+1)^k \rangle, \qquad k = 2, 3, 4,$$

to elaborate the non algorithmic approach to determine the polynomial Reps for degrees of syzygies in the Hilbert series $H\left(z, R_n^k\right)$, *the Frobenius numbers* $F\left(R_n^k\right)$ *and genera* $G\left(R_n^k\right)$.

For short, the results of LO'RS paper read,

$$F_j\left(R_n^2\right) = \sum_{i=1}^{3} A_i^j n^i, \ j = n \pmod 4, A_i^j \in \mathbb{Q}, \quad n \neq 3, 4, 5, 6, 9, 13,$$

$$F_j\left(R_n^3\right) = \sum_{i=1}^{5} B_i^j n^i, \ j = n \pmod{18}, \quad B_i^j \in \mathbb{Q}, \quad n \neq 3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 26, 27, 36, 45, 54, 63, 72, 90, 108, 126, 144, 162, 180, 198, 216, 234, 252, 270.$$

Observation: Excluding values of n give rise to the 4 symmetric, R_3^2, R_4^2, R_5^2 and R_3^3 and 30 semigroups nonsymmetric semigroups.

No more symmetric semigroups R_n^2 and R_n^3 do exist. Necessary conditions when a semigroup $\langle d_1, d_2, d_3 \rangle$ becomes symmetric:

$a)$ $d_3 \in \langle d_1, d_2 \rangle$, $\gcd(d_1, d_2) = 1$, $d_j > 3$, or (1)

$b)$ $\langle d_1, d_2, d_3 \rangle$, $d_j > 3$, satisfies Watanabe's Lemma (1973)

Lemma 1 (Watanabe, 1973)

Let a numerical semigroup $\langle d_1, d_2, d_3\rangle$ be given such that $d_j = a\delta_j$, $\delta_j \in \mathbb{Z}_+$, $j = 1, 2$ and $\gcd(\delta_1, \delta_2) = \gcd(a, d_3) = 1$. Then $\langle d_1, d_2, d_3\rangle$ is symmetric iff $d_3 \in \langle \delta_1, \delta_2\rangle$.

Symmetric numerical semigroups R_n^k

Proposition 1

There exist only three symmetric numerical semigroups R_n^2, $n = 3, 4, 5.$

Proof

Semigroups R_n^2, $n = 3, 4$, are symmetric due to requirement (1a). Find more n which satisfy (1a),

$$(n+1)^2 = a_1(n-1)^2 + a_2 n^2, \quad a_1, a_2 \in \mathbb{N}, \quad n > 4. \qquad (2)$$

Simplify (2) and obtain the Diophantine equation for $t = n - 4$,

$$(a_1 + a_2 - 1)t^2 + 2(3a_1 + 4a_2 - 5)t + 9a_1 + 16a_2 - 25 = 0,$$

with constraints (2) on three variables a_1, a_2, n. It has no solutions.

Consider another way to symmetrize R_n^2 by providing condition (1b), which may occur only when $n = 2p + 1$ and results in the Diophantine equation in b_1, b_2, p,

$$(2p+1)^2 = b_1 p^2 + b_2(p+1)^2, \quad b_1, b_2 \in \mathbb{N}, \quad p \geq 2, \qquad (3)$$

Solving (3) as a quadratic equation, we get

$$p = \frac{2 - b_2 \pm \Theta}{b_1 + b_2 - 4}, \quad \Theta^2 = b_1 + b_2 - b_1 b_2, \tag{4}$$

There exists only one appropriate solution, $b_1 = 4$, $b_2 = 1$, $p = 2$, which gives rise to semigroup R_5^2. □

Proposition 2

There exist only one symmetric numerical semigroup R_n^3, $n = 3$.

Proposition 3

There exist only three symmetric numerical semigroups R_n^4, $n = 3, 5, 7$.

Remark 1

An appearence of symmetric semigroups R_n^k, $n > 3, k > 4$, seems very rare. Numerical calculations give only two semigroups R_5^{11} and R_5^{13} among others R_{2p+1}^k, $2 \leq p \leq 50$, $5 \leq k \leq 10^3$.

$$\langle 4^{11}, 5^{11}, 6^{11} \rangle : \quad 5^{11} = 1093 \; 2^{11} + 263 \; 3^{11}$$
$$\langle 4^{13}, 5^{13}, 6^{13} \rangle : \quad 5^{13} = 51118 \; 2^{13} + 503 \; 3^{13}$$

Nonsymmetric semigroups $\langle d_1, d_2, d_3\rangle$

Let a nonsymmetric semigroup $S_3=\langle d_1,d_2,d_3\rangle$, $\gcd(d_1,d_2,d_3)=1$, $d_j\geq 3$ be given by matrix of minimal relations, $\mathbb{A}_3$, $a_{ij}\in\mathbb{Z}_+$,

$$\mathbb{A}_3\left(\begin{array}{c}d_1\\d_2\\d_3\end{array}\right)=\left(\begin{array}{c}0\\0\\0\end{array}\right),\quad \mathbb{A}_3=\left(\begin{array}{rrr}a_{11}&-a_{12}&-a_{13}\\-a_{21}&a_{22}&-a_{23}\\-a_{31}&-a_{32}&a_{33}\end{array}\right),\qquad (5)$$

$$\gcd(a_{11},a_{12},a_{13})=\gcd(a_{21},a_{22},a_{23})=\gcd(a_{31},a_{32},a_{33})=1.$$

$$\begin{aligned}a_{11}&=\min\left\{v_{11}\mid v_{11}\geq 2,\ v_{11}d_1=v_{12}d_2+v_{13}d_3,\ v_{12},v_{13}\in\mathbb{N}\cup\{0\}\right\},\\a_{22}&=\min\left\{v_{22}\mid v_{22}\geq 2,\ v_{22}d_2=v_{21}d_1+v_{23}d_3,\ v_{21},v_{23}\in\mathbb{N}\cup\{0\}\right\},\\a_{33}&=\min\left\{v_{33}\mid v_{33}\geq 2,\ v_{33}d_3=v_{31}d_1+v_{32}d_2,\ v_{31},v_{32}\in\mathbb{N}\cup\{0\}\right\}.\end{aligned}$$

All matrix elements a_{ij} are non-negative integers such that

$$a_{11}=a_{21}+a_{31},\quad a_{22}=a_{12}+a_{32},\quad a_{33}=a_{13}+a_{23},\quad a_{jj}\geq 2,\ a_{ij}\geq 1,$$
$$d_1=a_{22}a_{33}-a_{23}a_{32},\quad d_2=a_{33}a_{11}-a_{31}a_{13},\quad d_3=a_{11}a_{22}-a_{12}a_{21}.$$

The Hilbert series $H(z,S_3)$, the Frobenius number $F(S_3)$ and genus $G(S_3)$ are given by formulas

$$H(z,S_3)=\left(1-z^{a_{11}d_1}-z^{a_{22}d_2}-z^{a_{33}d_3}+z^{b_{11}}+z^{b_{22}}\right)\prod_{i=1}^{3}\left(1-z^{d_i}\right)^{-1},$$

$$\begin{aligned}&b_{11}=D_0+D_1,\quad b_{22}=D_0+D_2,\quad D_0=a_{11}a_{22}a_{33},\qquad (6)\\&D_1=a_{12}a_{23}a_{31},\quad D_2=a_{13}a_{32}a_{21},\quad D_3=d_1+d_2+d_3,\\&a_{11}d_1+a_{22}d_2+a_{33}d_3=b_{11}+b_{22}.\\&F_1=b_{11}-D_3,\quad F_2=b_{22}-D_3,\\&F(S_3)=\max\{F_1,F_2\},\qquad 2G(S_3)=1+D_0+D_1+D_2-D_3.\end{aligned}$$

Based on expression (6) we reduce almost all exclusive semigroups in LO'RS with $F(S_3)$ which differ from derived polynomials. An exclusion happens when in different ranges of n a difference $D_1(n) - D_2(n)$ may change its sign. In other words, the both sequences $F_1(n)$ and $F_2(n)$ contribute to the polynomial representation of $F(S_3)$.

Numerical semigroups R_n^2, $n \geq 6$

Write a minimal relation $\quad a_{33}(n+1)^2 = a_{32}n^2 + a_{31}(n-1)^2, \quad$ or

$$a_{32}n^2 = (a_{33} - a_{31})\left(n^2+1\right) + 2(a_{33} + a_{31})n. \tag{7}$$

Choose $a_{33} = a_{31}$ that results in $a_{32} = 4a_{33}/n$. The whole matrix $\mathbb{A}_3$ satisfies relations (5),

$$\left(\begin{array}{ccc} a_{21} + a_{33} & 4a_{33}/n - a_{22} & a_{23} - a_{33} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{33} & -4a_{33}/n & a_{33} \end{array}\right), \tag{8}$$

$$(n-1)^2 = a_{33}\left(a_{22} - \frac{4a_{23}}{n}\right),\; n^2 = a_{33}(a_{21}+a_{23}),\; (n+1)^2 = a_{33}\left(a_{22} + \frac{4a_{21}}{n}\right)$$

To provide entries in $\mathbb{A}_3$ be integers consider 4 cases, $n = j \pmod 4$.

1. $n = 4m$, $a_{33} = a_{31} = m$, $a_{32} = 1$.

$$\left(\begin{array}{ccc} a_{21} + m & 1 - a_{22} & a_{23} - m \\ -a_{21} & a_{22} & -a_{23} \\ -m & -1 & m \end{array}\right), \quad \begin{array}{rcl} (4m-1)^2 & = & a_{22}m - a_{23}, \\ (4m+1)^2 & = & a_{22}m + a_{21}. \end{array} \tag{9}$$

Two equations (9) allow to choose the following parameterization,

$a_{21}=km+1, \quad a_{22}=16m+8-k, \quad a_{23}=(16-k)m-1, \;\; 1 \le k \le 15,$

$$\left(\begin{array}{ccc} (k+1)m+1 & -[16m+7-k] & -[(k-15)m+1] \\ -(km+1) & 16m+8-k & -[(16-k)m-1] \\ -m & -1 & m \end{array}\right). \qquad (10)$$

Since $a_{13}, a_{23} \ge 1$ there is only one solution $k=15$, $m \ge 2$, that gives

$$\left(\begin{array}{ccc} 16m+1 & -8(2m-1) & -1 \\ -(15m+1) & 16m-7 & -(m-1) \\ -m & -1 & m \end{array}\right), \qquad (11)$$

$$G = 4m(34m^2-21m+2), \quad \begin{array}{rcll} F &=& 20, & m=1, \\ F &=& 272m^3-168m^2+m-2, & m \ge 2. \end{array}$$

2. $n=4m+2,$ $a_{33}=a_{31}=2m+1$, $a_{32}=2$.

$$\left(\begin{array}{ccc} 9m+5 & -(8m-1) & -(m+1) \\ -(7m+4) & 8m+1 & -m \\ -(2m+1) & -2 & 2m+1 \end{array}\right), \qquad (12)$$

$$G = m(80m^2+71m+16), \quad \begin{array}{rcll} F &=& 312, & m=1, \\ F &=& 160m^3+128m^2+10m-9, & m \ge 2. \end{array}$$

3. $n=4m+1,$ $a_{33}=a_{31}=4m+1$, $a_{32}=4$.

$$\left(\begin{array}{ccc} 7m+2 & -4(m-1) & -(3m+1) \\ -(3m+1) & 4m & -m \\ -(4m+1) & -4 & 4m+1 \end{array}\right), \qquad (13)$$

$$G = 2m(32m^2+9m+1), \quad \begin{array}{rcll} F &=& 112m^3+48m^2+8m-1, & m \le 3, \\ F &=& 128m^3-20m-5, & m \ge 4. \end{array}$$

4. $n = 4m+3$, $a_{33} = a_{31} = 4m+3$, $a_{32} = 4$.

$$\begin{pmatrix} 5m+4 & -4m & -(m+1) \\ -(m+1) & 4(m+1) & -(3m+2) \\ -(4m+3) & -4 & 4m+3 \end{pmatrix}, \qquad (14)$$

$$G = 2(32Nm^3 + 57m^2 + 33m + 6),$$
$$F = 128m^3 + 2242 + 124m + 19, \quad m \geq 1.$$

Numerical semigroups R_n^3, $n \geq 4$

Write a minimal relation

$$a_{32}n^3 = (a_{33} - a_{31})\left(n^3 + 3n\right) + (a_{33} + a_{31})(3n^2 + 1). \qquad (15)$$

Choose $a_{31} = (pn+q)$, $a_{33} = (pn-q)$, $p, q \in \mathbb{Q}$, and plug into (15),

$$a_{32}n^2 = 2p(3n^2+1) - 2q(n^2+3), \quad \text{or} \quad a_{32} = 6p - 2q + 2\frac{p-3q}{n^2}$$

To eliminate the dependence of a_{32} on n^{-2} in the last we put

$$p = 3q, \quad a_{31} = q(3n+1), \quad a_{32} = 16q, \quad a_{33} = q(3n-1). \qquad (16)$$

To satisfy $\gcd(a_{31}, a_{32}, a_{33}) = 1$ we have to distinguish two different cases: $q = 1$ if $n = 2N$ and $q = 1/2$ if $n = 2N+1$.

Numerical semigroups R_n^3, $n = 0 \pmod 2$

The matrix of minimal relations

$$\begin{pmatrix} a_{21}+6N+1 & 16-a_{22} & a_{23}-(6N-1) \\ -a_{21} & a_{22} & -a_{23} \\ -(6N+1) & -16 & (6N-1) \end{pmatrix}, \qquad (17)$$

$$(2N-1)^3 = a_{22}(6N-1) - 16a_{23}, \quad (2N+1)^3 = a_{22}(6N+1) + 16a_{21},$$

To balance the cubic degrees in (17) choose a_{2j} as polynomials on residue class of N modulo T_3, i.e., $N = T_3m + j$, $0 \le j < T_3$,

$$a_{21} = r_2m^2 + r_1m + r_0, \quad a_{22} = k_2m^2 + k_1m + k_0, \quad a_{23} = l_1m + l_0.$$

Substitute the above Reps into (17) and obtain

$$k_2 = r_2 = \frac{4T_3^2}{3}, \quad k_1 = \frac{8T_3}{9}(3j-2), \quad r_1 + l_1 = \frac{2T_3}{9}(1+12j),$$
$$3T_3k_0 + 4(r_1 - l_1) = \frac{T_3}{3}\left(9 + 16j + 12j^2\right),$$
$$(6j+1)k_0 + 16r_0 = (2j+1)^3, \quad (6j-1)k_0 - 16l_0 = (2j-1)^3,$$

that gives $T_3 = 9$, $k_2 = r_2 = 108$, $k_1 = 8(3j-2)$. Others parameters may be found numerically for nine first semigroups R^3_{18+2j}, $0 \le j \le 8$.

1. $n = 18m$, $m \ge 1$.

$$\begin{pmatrix} 108m^2+55m+1 & -(108m^2-16m-15) & -(53m-1) \\ -m(108m+1) & 108m^2-16m+1 & -m \\ -(54m+1) & -16 & 54m-1 \end{pmatrix},$$

$$G = 314928m^5 + 110808m^4 + 16632m^3 + 532m^2 - 62m,$$
$$F = 629856m^5 + 215784m^4 + 34020m^3 + 1890m^2 - 109m - 1, \ m \le 15,$$
$$F = 629856m^5 + 221616m^4 - 58320m^3 + 1944m^2 - 108m - 1, \ m \ge 16,$$

2. $n = 18m + 2$, $m \geq 1$; $n \neq 2$

$$\begin{pmatrix} 108m^2 + 37m + 3 & -(108m^2 + 8m - 3) & -(11m + 1) \\ -(108m^2 - 17m - 4) & 108m^2 + 8m + 13 & -(43m + 4) \\ -(54m + 7) & -16 & 54m + 5 \end{pmatrix},$$

$$G = 314928m^5 + 285768m^4 + 104760m^3 + 15244m^2 + 786m + 6,$$
$$F = 629856m^5 + 571536m^4 + 190512m^3 + 31752m^2 + 2548m + 75,$$

3. $n = 18m + 4$, $m \geq 1$; $n \neq 4.$

$$\begin{pmatrix} 108m^2 + 55m + 7 & -(108m^2 + 32m + 1) & -(5m + 1) \\ -(108m^2 + m - 6) & 108m^2 + 32m + 17 & -(49m + 10) \\ -(54m + 13) & -16 & 54m + 11 \end{pmatrix},$$

$$G = 314928m^5 + 460728m^4 + 270648m^3 + 77476m^2 + 10674m + 564,$$
$$F = 629856m^5 + 921456m^4 + 532656m^3 + 153144m^2 + 21812m + 1223,$$

4. $n = 18m + 6$, $m \geq 1$; $n \neq 6.$

$$\begin{pmatrix} 108m^2 + 109m + 25 & -(108m^2 + 56m - 3) & -(35m + 11) \\ -(108m^2 + 55m + 6) & 108m^2 + 56m + 13 & -(19m + 6) \\ -(54m + 19) & -16 & 54m + 17 \end{pmatrix},$$

$$G = 314928m^5 + 635688m^4 + 514296m^3 + 202780m^2 + 38434m + 2778,$$
$$F = 629856m^5 + 1271376m^4 + 968112m^3 + 355752m^2 + 63828m + 4499,$$

5. $n = 18m + 8$, $m \geq 0$,

$$\begin{pmatrix} 108m^2 + 145m + 44 & -(108m^2 + 80m + 1) & -(47m + 20) \\ -(108m^2 + 91m + 19) & 108m^2 + 80m + 17 & -(7m + 3) \\ -(54m + 25) & -16 & 54m + 23 \end{pmatrix},$$

$$G = 314928m^5 + 810648m^4 + 835704m^3 + 428308m^2 + 108658m + 10888,$$
$$F = 629856m^5 + 1580472m^4 + 1604772m^3 + 821178m^2 + 210979m + 21700,$$
$$m = 0, 1,$$
$$F = 629856m^5 + 1621296m^4 + 1590192m^3 + 753624m^2 + 173908m + 15695,$$
$$m \geq 2.$$

6. $n = 18m + 10$, $m \geq 0$,

$$\begin{pmatrix} 108m^2 + 163m + 58 & -(108m^2 + 104m + 13) & -(41m + 22) \\ -(108m^2 + 109m + 27) & 108m^2 + 104m + 29 & -(13m + 7) \\ -(54m + 31) & -16 & 54m + 29 \end{pmatrix},$$

$G = 314928m^5 + 985608m^4 + 1234872m^3 + 769612m^2 + 237666m + 29022,$

$F = 55222, \quad m = 0,$

$F = 629856m^5 + 1971216m^4 + 2398896m^3 + 1429704m^2 + 419252m + 48539,$

$m \geq 1,$

7. $n = 18m + 12$, $m \geq 0$,

$$\begin{pmatrix} 108m^2 + 163m + 61 & -(108m^2 + 128m + 33) & -(17m + 11) \\ -(108m^2 + 109m + 24) & 108m^2 + 128m + 49 & -(37m + 24) \\ -(54m + 37) & -16 & 54m + 35 \end{pmatrix},$$

$G = 314928m^5 + 1160568m^4 + 1711800m^3 + 1257796m^2 + 459058m + 66444,$

$F = 629856m^5 + 2321136m^4 + 3394224m^3 + 2466936m^2 + 892404m + 128663,$

8. $n = 18m + 14$, $m \geq 0$,

$$\begin{pmatrix} 108m^2 + 199m + 90 & -(108m^2 + 152m + 45) & -(29m + 22) \\ -(108m^2 + 145m + 47) & 108m^2 + 152m + 61 & -(25m + 19) \\ -(54m + 43) & -16 & 54m + 41 \end{pmatrix},$$

$G = 314928m^5 + 1335528m^4 + 2266488m^3 + 1917916m^2 + 807378m + 135042,$

$F = 629856m^5 + 2671056m^4 + 4482864m^3 + 3730536m^2 + 1541908m + 253539,$

9. $n = 18m + 16$, $m \geq 0$,

$$\begin{pmatrix} 108m^2 + 217m + 108 & -(108m^2 + 176m + 65) & -(23m + 20) \\ -(108m^2 + 163m + 59) & 108m^2 + 176m + 81 & -(31m + 27) \\ -(54m + 49) & -16 & 54m + 47 \end{pmatrix},$$

$G = 314928m^5 + 1510488m^4 + 2898936m^3 + 2776756m^2 + 1325266m + 251824,$

$F = 629856m^5 + 3020976m^4 + 5758128m^3 + 5458968m^2 + 2576660m + 484767,$

Numerical semigroups R_n^3, $n = 1 \pmod 2$

The matrix of minimal relations $\mathbb{A}_3$ reads,

$$\left(\begin{array}{ccc} a_{21}+(3N+2) & 8-a_{22} & a_{23}-(3N+1) \\ -a_{21} & a_{22} & -a_{23} \\ -(3N+2) & -8 & 3N+1 \end{array}\right),$$

$$(2N)^3 = a_{22}(3N+1) - 8a_{23}, \quad (2N+2)^3 = a_{22}(3N+2) + 8a_{21}.$$

1. $n = 18m+1$, $m \geq 0$,

$$\left(\begin{array}{ccc} 216m^2+29m+1 & -(216m^2-8m) & -m \\ -(216m^2+2m-1) & 216m^2-8m+8 & -(26m+1) \\ -(27m+2) & -8 & 27m+1 \end{array}\right),$$

$$G = 629856m^5 + 160380m^4 + 23220m^3 + 2330m^2 + 73m,$$
$$F = 1259712m^5 + 320760m^4 + 44712m^3 + 4644m^2 + 154m - 1,$$

2. $n = 18m+3$, $m \geq 1$; $\quad n \neq 3$.

$$\left(\begin{array}{ccc} 216m^2+83m+8 & -(216m^2+40m) & -(7m+1) \\ -(216m^2+56m+3) & 216m^2+40m+8 & -(20m+3) \\ -(27m+5) & -8 & 27m+4 \end{array}\right),$$

$$G = 629856m^5 + 510300m^4 + 172260m^3 + 30134m^2 + 2657m + 91,$$
$$F = 181, \quad m = 0,$$
$$F = 1259712m^5 + 1020600m^4 + 332424m^3 + 55404m^2 + 4698m + 157, \ m \geq 1.$$

3. $n = 18m+5$, $m \geq 0$,

$$\left(\begin{array}{ccc} 216m^2+128m+19 & -(216m^2+88m+8) & -(4m+1) \\ -(216m^2+101m+11) & 216m^2+88m+16 & -(23m+6) \\ -(27m+8) & -8 & 27m+7 \end{array}\right),$$

$$G = 629856m^5 + 860220m^4 + 476820m^3 + 134186m^2 + 18993m + 1066,$$
$$F = 1259712m^5 + 1720440m^4 + 946728m^3 + 263412m^2 + 37082m + 2107,$$

4. $n = 18m + 7$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 191m + 42 & -(216m^2 + 136m + 16) & -(19m + 7) \\ -(216m^2 + 164m + 31) & 216m^2 + 136m + 24 & -(8m + 3) \\ -(27m + 11) & -8 & 27m + 10 \end{pmatrix},$$

$G = 629856m^5 + 1210140m^4 + 936900m^3 + 365246m^2 + 71609m + 5637$,
$F = 10745, \quad m = 0$,
$F = 1259712m^5 + 2420280m^4 + 1840968m^3 + 693468m^2 + 129322m + 9537$,
$m \geq 1$,

5. $n = 18m + 9$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 245m + 69 & -(216m^2 + 184m + 32) & -(25m + 12) \\ -(216m^2 + 218m + 55) & 216m^2 + 184m + 40 & -(2m + 1) \\ -(27m + 14) & -8 & 27m + 13 \end{pmatrix},$$

$G = 629856m^5 + 1560060m^4 + 1552500m^3 + 776234m^2 + 195001m + 19684$,
$F = 1259712m^5 + 3108456m^4 + 3083184m^3 + 1537596m^2 + 385666m + 38919$,
$m \leq 3$,
$F = 1259712m^5 + 3120120m^4 + 3061800m^3 + 1488132m^2 + 358074m + 34087$,
$m \geq 4$,

6. $n = 18m + 11$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 290m + 97 & -(216m^2 + 232m + 56) & -(22m + 13) \\ -(216m^2 + 263m + 80) & 216m^2 + 232m + 64 & -(5m + 3) \\ -(27m + 17) & -8 & 27m + 16 \end{pmatrix},$$

$G = 629856m^5 + 1909980m^4 + 2323620m^3 + 1417694m^2 + 433745m + 53223$,
$F = 103589, \quad m = 0$,
$F = 1259712m^5 + 3819960m^4 + 4609224m^3 + 2766636m^2 + 826058m + 98125$,
$m \geq 1$,

7. $n = 18m + 13$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 326m + 123 & -(216m^2 + 280m + 90) & -(10m + 7) \\ -(216m^2 + 299m + 103) & 216m^2 + 280m + 96 & -(17m + 12) \\ -(27m + 20) & -8 & 27m + 19 \end{pmatrix},$$

$$G = 629856m^5 + 2259900m^4 + 3250260m^3 + 2342114m^2 + 845465m + 122286,$$
$$F = 1259712m^5 + 3120120m^4 + 3061800m^3 + 1488132m^2 + 358074m + 34087.$$

8. $n = 18m + 15$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 380m + 167 & -(216m^2 + 328m + 120) & -(16m + 13) \\ -(216m^2 + 353m + 144) & 216m^2 + 328m + 128 & -(11m + 9) \\ -(27m + 23) & -8 & 27m + 22 \end{pmatrix},$$

$$G = 629856m^5 + 2609820m^4 + 4332420m^3 + 3601550m^2 + 1499153m + 249937,$$
$$F = 1259712m^5 + 5219640m^4 + 8637192m^3 + 7135452m^2 + 2943162m + 484897.$$

9. $n = 18m + 17$, $m \geq 0$,

$$\begin{pmatrix} 216m^2 + 425m + 209 & -(216m^2 + 376m + 160) & -(13m + 12) \\ -(216m^2 + 398m + 183) & 216m^2 + 376m + 168 & -(14m + 13) \\ -(27m + 26) & -8 & 27m + 25 \end{pmatrix},$$

$$G = 629856m^5 + 2959740m^4 + 5570100m^3 + 5247626m^2 + 2474713m + 467304,$$
$$F = 1259712m^5 + 5919480m^4 + 11117736m^3 + 10433124m^2 + 4892186m + 917039.$$

Exceptional numerical semigroups R_n^3, $n = 4, 6$

$$R_4^3 : \begin{pmatrix} 7 & -1 & -1 \\ -1 & 18 & -9 \\ -6 & -17 & 10 \end{pmatrix}, \quad \begin{matrix} G = 558 \\ F = 1098 \end{matrix}, \qquad R_6^3 : \begin{pmatrix} 31 & -10 & -5 \\ -6 & 13 & -6 \\ -25 & -3 & 11 \end{pmatrix}, \quad \begin{matrix} G = 2670 \\ F = 5249 \end{matrix}$$

Numerical semigroups R_n^4, $n \geq 4$, $n = 0 \pmod 2$

1. $n = 40m$, $m \geq 4$; $n \neq 40, 80, 120.$

$$\begin{pmatrix} 8160m^2+161m+1 & -(320m^2-1280m+1) & -(7840m^2-159m+1) \\ -(160m^2+m) & 320m^2+1 & -(160m^2-m) \\ -(8000m^2+160m+1) & -1280m & 8000m^2-160m+1 \end{pmatrix}$$

2. $n = 40m+2$, $m \geq 1$; $n \neq 2.$

$$\begin{pmatrix} 4640m^2+526m+15 & -(4480m^2+64m+6) & -(160m^2-18m-1) \\ -(4400m^2+495m+14) & 4800m^2+160m+11 & -(400m^2+65m+2) \\ -(240m^2+31m+1) & -(320m^2+96m+5) & 560m^2+47m+1 \end{pmatrix}$$

3. $n = 40m+4$, $m \geq 0.$

$$\begin{pmatrix} 2880m^2+616m+33 & -8(320m^2+32m+1) & -(320m^2+40m+1) \\ -(2240m^2+473m+25) & 2880m^2+448m+25 & -(640m^2+135m+7) \\ -(640m^2+143m+8) & -(320m^2+192m+17) & 960m^2+175m+8 \end{pmatrix}$$

4. $n = 40m+6$, $m \geq 1$; $n \neq 6.$

$$\begin{pmatrix} 2800m^2+885m+70 & -(1600m^2+160m-7) & -(1200m^2+325m+22) \\ -(1760m^2+550m+43) & 1920m^2+448m+30 & -(160m^2+58m+5) \\ -(1040m^2+335m+27) & -(320m^2+288m+37) & 1360m^2+383m+27 \end{pmatrix}$$

5. $n = 40m+8$, $m \geq 0.$

$$\begin{pmatrix} 2240m^2+932m+97 & -4(320m^2+64m+1) & -(960m^2+356m+33) \\ -(800m^2+325m+33) & 1600m^2+640m+69 & -(800m^2+315m+31) \\ -(1440m^2+607m+64) & -(320m^2+384m+65) & 1760m^2+671m+64 \end{pmatrix}$$

6. $n = 40m+10$, $m \geq 1$; $n \neq 10.$

$$\begin{pmatrix} 2480m^2+1283m+166 & -(960m^2+160m-17) & -(1520m^2+723m+86) \\ -(640m^2+324m+41) & 1280m^2+640m+84 & -(640m^2+316m+39) \\ -(1840m^2+959m+125) & -(320m^2+480m+101) & 2160m^2+1039m+125 \end{pmatrix}$$

7. $n = 40m + 12$, $m \geq 0$.

$$\begin{pmatrix} 1280m^2 + 787m + 121 & -(960m^2 + 448m + 51) & -(320m^2 + 179m + 25) \\ -(320m^2 + 183m + 26) & 2240m^2 + 1472m + 247 & -(1920m^2 + 1129m + 166) \\ -(960m^2 + 604m + 95) & -(1280m^2 + 1024m + 196) & 2240m^2 + 1308m + 191 \end{pmatrix}$$

8. $n = 40m + 14$, $m \geq 1$; $n \neq 14$.

$$\begin{pmatrix} 2720m^2 + 1954m + 351 & -(640m^2 + 64m - 54) & -(2080m^2 + 1410m + 239) \\ -(80m^2 + 51m + 8) & 960m^2 + 736m + 143 & -(880m^2 + 605m + 104) \\ -(2640m^2 + 1903m + 343) & -(320m^2 + 672m + 197) & 2960m^2 + 2015m + 343 \end{pmatrix}$$

9. $n = 40m + 16$, $m \geq 0$.

$$\begin{pmatrix} 1920m^2 + 1570m + 321 & -(640m^2 + 256m + 2) & -(1280m^2 + 994m + 193) \\ -(800m^2 + 645m + 130) & 1600m^2 + 1280m + 261 & -(800m^2 + 635m + 126) \\ -(1120m^2 + 925m + 191) & -(960m^2 + 1024m + 259) & 2080m^2 + 1629m + 319 \end{pmatrix}$$

10. $n = 40m + 18$, $m \geq 0$.

$$\begin{pmatrix} 1120m^2 + 1026m + 235 & -(640m^2 + 448m + 74) & -(480m^2 + 418m + 91) \\ -(1040m^2 + 937m + 211) & 2880m^2 + 2656m + 621 & -(1840m^2 + 1639m + 365) \\ -(80m^2 + 89m + 24) & -(2240m^2 + 2208m + 547) & 2320m^2 + 2057m + 456 \end{pmatrix}$$

11. $n = 40m + 20$, $m \geq 2$; $n \neq 60$.

$$\begin{pmatrix} 4160m^2 + 4241m + 1081 & -(320m^2 - 320m - 239) & -(3840m^2 + 3761m + 921) \\ -(320m^2 + 322m + 81) & 640m^2 + 640m + 162 & -(320m^2 + 318m + 79) \\ -(3840m^2 + 3919m + 1000) & -(320m^2 + 960m + 401) & 4160m^2 + 4079m + 1000 \end{pmatrix}$$

12. $n = 40m - 18$, $m \geq 1$.

$$\begin{pmatrix} 2320m^2 - 2057m + 456 & -(2240m^2 - 2208m + 547) & -(80m^2 - 89m + 24) \\ -(1840m^2 - 1639m + 365) & 2880m^2 - 2656m + 621 & -(1040m^2 - 937m + 211) \\ -(480m^2 - 418m + 91) & -(640m^2 - 448m + 74) & 1120m^2 - 1026m + 235 \end{pmatrix}$$

13. $n = 40m - 16$, $m \geq 1$.

$$\begin{pmatrix} 2080m^2 - 1629m + 319 & -(960m^2 - 1024m + 259) & -(1120m^2 - 925m + 191) \\ -(800m^2 - 635m + 126) & 1600m^2 - 1280m + 261 & -(800m^2 - 645m + 130) \\ -(1280m^2 - 994m + 193) & -(640m^2 - 256m + 2) & 1920m^2 - 1570m + 321 \end{pmatrix}$$

14. $n = 40m - 14$, $m \geq 2$; $n \neq 26$.

$$\begin{pmatrix} 2960m^2 - 2015m + 343 & -(320m^2 - 672m + 197) & -(2640m^2 - 1903m + 343) \\ -(880m^2 - 605m + 104) & 960m^2 - 736m + 143 & -(80m^2 - 51m + 8) \\ -(2080m^2 - 1410m + 239) & -(640m^2 - 64m - 54) & 2720m^2 - 1954m + 351 \end{pmatrix}$$

15. $n = 40m - 12$, $m \geq 1$.

$$\begin{pmatrix} 2240m^2 - 1308m + 191 & -4(320m^2 - 256m + 49) & -(960m^2 - 604m + 95) \\ -(1920m^2 - 1129m + 166) & 2240m^2 - 1472m + 247 & -(320m^2 - 183m + 26) \\ -(320m^2 - 179m + 25) & -(960m^2 - 448m + 51) & 1280m^2 - 787m + 121 \end{pmatrix}$$

16. $n = 40m - 10$, $m \geq 2$; $n \neq 30$.

$$\begin{pmatrix} 2160m^2 - 1039m + 125 & -(320m^2 - 480m + 101) & -(1840m^2 - 959m + 125) \\ -(640m^2 - 316m + 39) & 1280m^2 - 640m + 84 & -(640m^2 - 324m + 41) \\ -(1520m^2 - 723m + 86) & -(960m^2 - 160m - 17) & 2480m^2 - 1283m + 166 \end{pmatrix}$$

17. $n = 40m - 8$, $m \geq 1$.

$$\begin{pmatrix} 1760m^2 - 671m + 64 & -(320m^2 - 384m + 65) & -(1440m^2 - 607m + 64) \\ -(800m^2 - 315m + 31) & 1600m^2 - 640m + 69 & -(800m^2 - 325m + 33) \\ -(960m^2 - 356m + 33) & -(1280m^2 - 256m + 4) & 2240m^2 - 932m + 97 \end{pmatrix}$$

18. $n = 40m - 6$, $m \geq 1$.

$$\begin{pmatrix} 1360m^2 - 383m + 27 & -(320m^2 - 288m + 37) & -(1040m^2 - 335m + 27) \\ -(160m^2 - 58m + 5) & 1920m^2 - 448m + 30 & -(1760m^2 - 550m + 43) \\ -(1200m^2 - 325m + 22) & -(1600m^2 - 160m - 7) & 2800m^2 - 885m + 70 \end{pmatrix}$$

19. $n = 40m - 4$, $m \geq 1$.

$$\begin{pmatrix} 960m^2 - 175m + 8 & -(320m^2 - 192m + 17) & -(640m^2 - 143m + 8) \\ -(640m^2 - 135m + 7) & 2880m^2 - 448m + 25 & -(2240m^2 - 473m + 25) \\ -(320m^2 - 40m + 1) & -(2560m^2 - 256m + 8) & 2880m^2 - 616m + 33 \end{pmatrix}$$

20. $n = 40m - 2$, $m \geq 1$.

$$\begin{pmatrix} 560m^2 - 47m + 1 & -(320m^2 - 96m + 5) & -(240m^2 - 31m + 1) \\ -(400m^2 - 65m + 2) & 4800m^2 - 160m + 11 & -(4400m^2 - 495m + 14) \\ -(160m^2 + 18m - 1) & -(4480m^2 - 64m + 6) & 4640m^2 - 526m + 15 \end{pmatrix}$$

Numerical semigroups R_n^4, $n \geq 4$, $n = 1 \pmod 2$

1. $n = 20m+1$, $m \geq 1$.

$$\begin{pmatrix} 230m^2+30m+1 & -32m(5m-1) & -2m(35m+1) \\ -10m(15m+1) & 16(50m^2+10m+1) & -(650m^2+50m+1) \\ -(80m^2+20m+1) & -16(40m^2+12m+1) & 720m^2+52m+1 \end{pmatrix}$$

2. $n = 20m+3$, $m \geq 1$; $n \neq 3$.

$$\begin{pmatrix} 530m^2+178m+15 & -16(10m^2-6m-1) & -(370m^2+94m+6) \\ -(60m^2+16m+1) & 16(20m^2+8m+1) & -(260m^2+72m+5) \\ -(470m^2+162m+14) & -32(5m^2+7m+1) & 630m^2+166m+11 \end{pmatrix}$$

3. $n = 20m+5$, $m \geq 1$; $n \neq 5$.

$$\begin{pmatrix} 490m^2+258m+34 & -16(30m^2+10m+1) & -(10m^2-2m-1) \\ -(320m^2+164m+21) & 16(40m^2+20m+3) & -(320m^2+156m+19) \\ -(170m^2+94m+13) & -32(5m^2+5m+1) & 330m^2+154m+18 \end{pmatrix}$$

4. $n = 20m+7$, $m \geq 2$; $n \neq 7, 27$.

$$\begin{pmatrix} 2(565m^2+417m+77) & -32(5m^2-7m-3) & -(970m^2+638m+105) \\ -(130m^2+94m+17) & 16(10m^2+6m+1) & -(30m^2+22m+4) \\ -(1000m^2+740m+137) & -16(20m+7) & 1000m^2+660m+109 \end{pmatrix}$$

5. $n = 20m+9$, $m \geq 1$; $n \neq 9$.

$$\begin{pmatrix} 430m^2+402m+94 & -16(10m^2+2m-1) & -(270m^2+230m+49) \\ -(290m^2+266m+61) & 32(15m^2+13m+3) & -(190m^2+170m+38) \\ -(140m^2+136m+33) & -16(20m^2+24m+7) & 460m^2+400m+87 \end{pmatrix}$$

6. $n = 20m-9$, $m \geq 1$.

$$\begin{pmatrix} 460m^2-400m+87 & -16(20m^2-24m+7) & -(140m^2-136m+33) \\ -(190m^2-170m+38) & 32(15m^2-13m+3) & -(290m^2-266m+61) \\ -(270m^2-230m+49) & -16(10m^2-2m-1) & 430m^2-402m+94 \end{pmatrix}$$

7. $n = 20m - 7$, $m \geq 3$; $n \neq 13, 33.$

$$\begin{pmatrix} 1030m^2 - 682m + 113 & -32(5m^2 - 13m + 4) & -(870m^2 - 646m + 120) \\ -(30m^2 - 22m + 4) & 16(10m^2 - 6m + 1) & -(130m^2 - 94m + 17) \\ -(1000m^2 - 660m + 109) & -16(20m - 7) & 1000m^2 - 740m + 137 \end{pmatrix}$$

8. $n = 20m - 5$, $m \geq 1$.

$$\begin{pmatrix} 330m^2 - 154m + 18 & -32(5m^2 - 5m + 1) & -(170m^2 - 94m + 13) \\ -(320m^2 - 156m + 19) & 16(40m^2 - 20m + 3) & -(320m^2 - 164m + 21) \\ -(10m^2 + 2m - 1) & -16(30m^2 - 10m + 1) & 490m^2 - 258m + 34 \end{pmatrix}$$

9. $n = 20m - 3$, $m \geq 2$; $n \neq 17.$

$$\begin{pmatrix} 630m^2 - 166m + 11 & -32(5m^2 - 7m + 1) & -(470m^2 - 162m + 14) \\ -(260m^2 - 72m + 5) & 16(20m^2 - 8m + 1) & -(60m^2 - 16m + 1) \\ -(370m^2 - 94m + 6) & -16(10m^2 + 6m - 1) & 530m^2 - 178m + 15 \end{pmatrix}$$

10. $n = 20m - 1$, $m \geq 1$.

$$\begin{pmatrix} 720m^2 - 52m + 1 & -16(40m^2 - 12m + 1) & -(80m^2 - 20m + 1) \\ -(650m^2 - 50m + 1) & 16(50m^2 - 10m + 1) & -10m(15m - 1) \\ -2m(35m - 1) & -32m(5m + 1) & 230m^2 - 30m + 1 \end{pmatrix}$$

Exceptional numerical semigroups R_n^4

$$R_5^4 : \begin{pmatrix} 81 & 0 & -16 \\ -34 & 16 & -1 \\ -81 & 0 & 16 \end{pmatrix}, \quad \begin{matrix} G = 14280 \\ F = 28559 \end{matrix},$$

$$R_7^4 : \begin{pmatrix} 256 & 0 & -81 \\ -17 & 16 & -4 \\ -256 & 0 & 81 \end{pmatrix}, \quad \begin{matrix} G = 181200, \\ F = 362399 \end{matrix}$$

$$R_6^4 : \begin{pmatrix} 113 & -23 & -17 \\ -43 & 30 & -5 \\ -70 & -7 & 22 \end{pmatrix}, \quad \begin{matrix} G = 41713, \\ F = 78308. \end{matrix}$$

n /=/ 3, 5, 6, 7, 9, 10, 13, 14, 17, 20, 26, 27, 30, 33, 40, 60, 80, 120

$$R_9^4 : \begin{pmatrix} 155 & -80 & -11 \\ -61 & 96 & -38 \\ -94 & -16 & 49 \end{pmatrix}, \qquad \begin{array}{l} G = 502480, \\ F = 994223. \end{array}$$

$$R_{10}^4 : \begin{pmatrix} 207 & -67 & -47 \\ -41 & 84 & -39 \\ -166 & -17 & 86 \end{pmatrix}, \qquad \begin{array}{l} G = 965342, \\ F = 1897924. \end{array}$$

$$R_{13}^4 : \begin{pmatrix} 485 & -32 & -238 \\ -12 & 80 & -53 \\ -473 & -48 & 291 \end{pmatrix}, \qquad \begin{array}{l} G = 6071192, \\ F = 12005295. \end{array}$$

$$R_{14}^4 : \begin{pmatrix} 359 & -89 & -135 \\ -8 & 143 & -104 \\ -351 & -54 & 239 \end{pmatrix}, \qquad \begin{array}{l} G = 7729559, \\ F = 15400797. \end{array}$$

$$R_{17}^4 : \begin{pmatrix} 668 & -176 & -277 \\ -193 & 208 & -45 \\ -475 & -32 & 322 \end{pmatrix}, \qquad \begin{array}{l} G = 24979344, \\ F = 48247935. \end{array}$$

$$R_{20}^4 : \begin{pmatrix} 1243 & -85 & -763 \\ -81 & 162 & -79 \\ -1162 & -77 & 842 \end{pmatrix}, \qquad \begin{array}{l} G = 90813516, \\ F = 176868200. \end{array}$$

$$R_{26}^4 : \begin{pmatrix} 1667 & -212 & -1043 \\ -379 & 367 & -37 \\ -1288 & -155 & 1080 \end{pmatrix}, \qquad \begin{array}{l} G = 365363593, \\ F = 720624113. \end{array}$$

$$R_{27}^4 : \begin{pmatrix} 2359 & -112 & -1657 \\ -241 & 272 & -561 \\ -2118 & -160 & 1713 \end{pmatrix}, \qquad \begin{array}{l} G = 647256024, \\ F = 1230618127. \end{array}$$

$$R_{30}^4 : \begin{pmatrix} 1609 & -665 & -649 \\ -363 & 724 & -357 \\ -1246 & -59 & 1006 \end{pmatrix}, \qquad \begin{array}{l} G = 739585479, \\ F = 1465271324. \end{array}$$

$$R^4_{33}: \begin{pmatrix} 2949 & -400 & -1959 \\ -80 & 464 & -349 \\ -2869 & -64 & 2308 \end{pmatrix}, \quad \begin{array}{l} G = 1782545568, \\ F = 3555061055. \end{array}$$

$$R^4_{40}: \begin{pmatrix} 8805 & -4 & -7205 \\ -161 & 321 & -159 \\ -8644 & -317 & 7364 \end{pmatrix}, \quad \begin{array}{l} G = 10589583194, \\ F = 21173668803. \end{array}$$

$$R^4_{60}: \begin{pmatrix} 10205 & -1203 & -7805 \\ -723 & 1442 & -717 \\ -9482 & -239 & 8522 \end{pmatrix}, \quad \begin{array}{l} G = 67447447193, \\ F = 133546213800. \end{array}$$

$$R^4_{80}: \begin{pmatrix} 33605 & -2 & -30405 \\ -642 & 1281 & -638 \\ -32963 & -1279 & 31043 \end{pmatrix}, \quad \begin{array}{l} G = 680612207996, \\ F = 1361182355203. \end{array}$$

$$R^4_{120}: \begin{pmatrix} 75367 & -1922 & -68647 \\ -1443 & 2881 & -1437 \\ -73924 & -959 & 70084 \end{pmatrix}, \quad \begin{array}{l} G = 7758023870871, \\ F = 15421051483202. \end{array}$$

Duality of numerical semigroups $R^4_{T_4 m+j}$ and $R^4_{T_4 m-j}$, $T_4 = 20, 40$

Example 1. Semigroups R^4_{20m+9} and R^4_{20m-9}

$$\begin{aligned} G_+ &= 8(7766000m^6 + 21049600m^5 + 23809780m^4 + 14385560m^3 + \\ &\quad 4895936m^2 + 889781m + 67446), \\ F_+ &= 120(10m)^6 + 3280(10m)^5 + 37424(10m)^4 + 228128(10m)^3 + \\ &\quad 783448(10m)^2 + 14368816m + 1099199 \\ G_- &= 8(7766000m^6 - 21049600m^5 + 23809780m^4 - 14385560m^3 + \\ &\quad 4895936m^2 - 889781m + 67446), \\ F_- &= 120(10m)^6 - 3280(10m)^5 + 37424(10m)^4 - 228128(10m)^3 + \\ &\quad 783448(10m)^2 - 14368816m + 1099199 \end{aligned}$$

Example 2. Semigroups R^4_{40m+2} and R^4_{40m-2}

$$\begin{aligned}
G_+ &= 6563840000m^6 + 1537792000m^5 + 149748800m^4 + 7837120m^3 + \\
&\quad 226429m^2 + 2996m + 5, \\
F_+ &= 12902400000m^6 + 3008000000m^5 + 299520000m^4 + 16976000m^3 + \\
&\quad 584400m^2 + 11071m + 79 \\
G_- &= 6563840000m^6 - 1537792000m^5 + 149748800m^4 - 7837120m^3 + \\
&\quad 226429m^2 - 2996m + 5, \\
F_- &= 12902400000m^6 - 3008000000m^5 + 299520000m^4 - 16976000m^3 + \\
&\quad 584400m^2 - 11071m + 79
\end{aligned}$$

Theorem 1.

Let two numerical semigroups be given by their minimal relations,

$$R^4_{T_4m\pm k} : \begin{pmatrix} E^{\pm}_{11}(m) & -E^{\pm}_{12}(m) & -E^{\pm}_{13}(m) \\ -E^{\pm}_{21}(m) & E^{\pm}_{22}(m) & -E^{\pm}_{23}(m) \\ -E^{\pm}_{31}(m) & -E^{\pm}_{32}(m) & E^{\pm}_{33}(m) \end{pmatrix}, \quad k \le T_4/2,$$

where $E^-_{ij}(m)$ and $E^+_{ij}(m)$ are given by polynomials

$E^-_{ij}(m) = A^-_{ij}m^2 - B^-_{ij}m + C^-_{ij}, \qquad E^+_{ij}(m) = A^+_{ij}m^2 + B^+_{ij}m + C^+_{ij}.$

Then the following duality relations hold ($W_{ij} = A_{ij} = B_{ij} = C_{ij}$),

$$\begin{array}{lll} W^+_{11} = W^-_{33}, & W^+_{12} = W^-_{32}, & W^+_{13} = W^-_{31}, \\ W^+_{21} = W^-_{23}, & W^+_{22} = W^-_{22}, & W^+_{23} = W^-_{21}, \\ W^+_{31} = W^-_{13}, & W^+_{32} = W^-_{12}, & W^+_{33} = W^-_{11}. \end{array}$$

Theorem 2.

Let genera $G^{\pm}$ of two dual semigroups $R^4_{T_4m\pm k}$ are given by

$$\begin{aligned}
R^4_{T_4m-k} &: \; G^-(m) = g^-_6 m^6 + g^-_5 m^5 + g^-_4 m^4 + g^-_3 m^3 + g^-_2 m^2 + g^-_1 m + g^-_0, \\
R^4_{T_4m+k} &: \; G^+(m) = g^+_6 m^6 + g^+_5 m^5 + g^+_4 m^4 + g^+_3 m^3 + g^+_2 m^2 + g^+_1 m + g^+_0,
\end{aligned}$$

Then the following duality relations hold

$$g_{2k}^- = g_{2k}^+, \qquad g_{2k+1}^- = -g_{2k+1}^+, \quad k = 0, 1, 2, 3.$$

Conjecture and question

Conjecture 1.

Let a numerical semigroup R_n^k, $n = T_k m + j$, be given by their minimal relations on residue class of n modulo T_k,

$$R_{T_k m + j}^k : \begin{pmatrix} K_{11}^{(k)}(m) & -K_{12}^{(k)}(m) & -K_{13}^{(k)}(m) \\ -K_{21}^{(k)}(m) & K_{22}^{(k)}(m) & -K_{23}^{(k)}(m) \\ -K_{31}^{(k)}(m) & -K_{32}^{(k)}(m) & K_{33}^{(k)}(m) \end{pmatrix}, \quad j \le T_k/2.$$

Then polynomial expressions for $K_{ij}^{(k)}(m)$ read in two different cases.
If $k = 2q$, then

$$K_{ij}^{(2q)}(m) = A_{ij}m^q + B_{ij}m^{q-1} + \ldots + C_{ij}m + D_{ij}, \quad 1 \le i, j \le 3,$$
$$F(n), G(n) = \mathcal{O}\left(n^{3q}\right).$$

If $k = 2q + 1$, then matrix elements $(i, j) = (1, 1), (1, 2), (2, 1), (2, 2)$ are given by

$$K_{ij}^{(2q+1)}(m) = E_{ij}m^{q+1} + I_{ij}m^q + \ldots + J_{ij}m + H_{ij},$$

while the matrix elements with $(i, j) = (1, 3), (2, 3), (3, 1), (3, 2), (3, 3)$ are given by

$$K_{ij}^{(2q+1)}(m) = M_{ij}m^q + N_{ij}m^{q-1} + \ldots + P_{ij}m + S_{ij},$$
$$F(n), G(n) = \mathcal{O}\left(n^{3q+2}\right).$$

Question 1.

Keeping in mind $T_2 = 4,\ T_3 = 18,\ T_4 = 40$, find T_k for the higher k.

THANK YOU FOR ATTENTION

LA RINGRAZIO PER L'ATTENZIONE