Some new results on Wilf's conjecture

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$g \backslash q$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
21	1	17710	34069	7943	1750	453	172	46	19	15	9	2	2	2	0
22	1	28656	57566	13108	2806	707	249	81	32	16	16	2	2	2	1
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Thus:

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Grazie mille per la sua attenzione :-)