On a question of Eliahou and a conjecture of Wilf

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Outline

Motivation

This work is motivated by a recent paper of Eliahou:

S. Eliahou, Wilf's conjecture and Macaulay's theorem, preprint.

It was presented in a special session on "Commutative Monoids" of the AMS-EMS-SPM joint meeting that held in Porto, in 2015.

Many talks in former editions of the IMNS series of conferences on numerical semigroups referred Wilf's conjecture...

A pictorial view of a numerical semigroup

We represent a numerical semigroup with multiplicity m and conductor c by highlighting some entries in a $(\lceil c/m \rceil + 1) \times m$ table of consecutive integers.

The uppermost row starts in c. The lowest row contains 0.

52	53	54	55	56	57	58	59	60	61	62
41	42	43	44	45	46	47	48	49	50	51
30	31	32	33	34	35	36	37	38	39	40
19	20	21	22	23	24	25	26	27	28	29
8	9	10	11	12	13	14	15	16	17	18
-3	-2	-1	0	1	2	3	4	5	6	7

Pictorial representation of the numerical semigroup $\langle 11, 13, 21, 62 \rangle$

Among the elements of the numerical semigroup, some are highlighted.

Terminology, notation and a conjecture of Wilf

Let S be a numerical semigroup.

Primitive elements of S (also known as minimal generators) .... $\mathrm{P}(S)$

Conductor of S .... $\mathrm{c}(S)$

the smallest integer from which all the integers belong to S

Lower elements of S (also known as left elements) .... $\mathrm{L}(S)$

the elements of S that are smaller than $\mathrm{c}(S)$

Wilf number .... $\mathrm{W}(S) = |P||L| - c$

Conjecture (Wilf)

If S is a numerical semigroup, then $\mathrm{W}(S) \geq 0$.

Some more notation

Multiplicity of S .... $\mathrm{m}(S)$ the smallest positive element of S

q-**number** .... $q(S) = \lceil \mathrm{c} / \mathrm{m} \rceil$

The non primitive elements are called *decomposable*.

Decomposable elements at level q .... $D_q(S) = [c, c+m[\setminus P$.

A kind of a remainder .... $\rho(S) = q \cdot \mathrm{m} - \mathrm{c}$.

Eliahou number .... $\mathrm{E}(S) = |P \cap L||L| - q|D_q| + \rho$

Proposition (Eliahou)

If $\mathrm{E}(S) \geq 0$, *then* $\mathrm{W}(S) \geq 0$.

Theorem (Eliahou)

Numerical semigroups S *satisfying* $q(S) = 3$ *have positive Eliahou number.*

A problem of Eliahou

Most numerical semigroups satisfy $q(S) \leq 3$... A natural question is:

"Is there any numerical semigroup S such that $\mathrm{E}(S) < 0$?"

The answer is "yes". In fact, there are infinitely many, but...

For instance, the probability of a numerical semigroup taken at random from the set of numerical semigroups of genus up to 60 is $\simeq \frac{5}{10^{13}}$.

So, without computational tools, we would possibly never find a single example.

Problem (Eliahou)

Characterize the numerical semigroups having negative Eliahou number.

$\langle \mathbf{m}, \mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_r} \rangle_c$ is the numerical semigroup containing $\{m, g_1, g_2, \ldots, g_r\}$ and all the integers greater than or equal to c.

Fromentin-Eliahou examples . . . notation . . . "generators"

56	57	58	59	60	61	62	63	64	65	66	67	68	69
42	43	44	45	46	47	48	49	50	51	52	53	54	55
28	29	30	31	32	33	34	35	36	37	38	39	40	41
14	15	16	17	18	19	20	21	22	23	24	25	26	27
0	1	2	3	4	5	6	7	8	9	10	11	12	13

. . . . $S(4) \ldots \langle 14, 22, 23 \rangle_{56}$

64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

. . . . $S^{(1)}(4, 0) \ldots \langle 16, 25, 26 \rangle_{64}$

68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84
51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67
34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

. . . . $\langle 17, 26, 28 \rangle_{68}$

68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84
51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67
34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

. . . . $\langle 17, 27, 28 \rangle_{68}$

72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89
54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71
36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

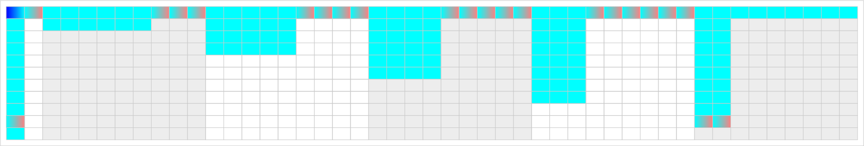
. . . . . $S^{(2)}(4, 0) \ldots \langle 18, 28, 29 \rangle_{72}$

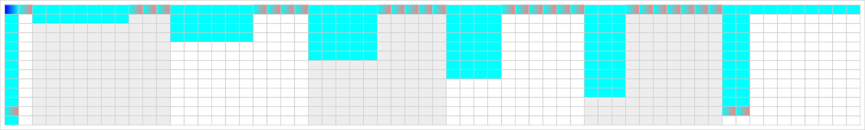
The Fromentin-Eliahou examples have Eliahou number equal to -1 and are the only examples of numerical semigroups of genus not greater than 60 having negative Eliahou number.

Defining the numerical semigroup $S(p)$

A visual way to look to $S(p)$

We construct numerical semigroups whith pictorial representations similar to those of some of the Fromentin-Eliahou examples.





One can partition these tables into several blocks (where p is an even positive integer):

a block C consisting of the 2 leftmost columns;

blocks B_i, $1 \leq i \leq \frac{p}{2}$, consisting of $\frac{p}{2} + 4$ consecutive columns.

The definition of $S(p)$

Let p be an **even** positive integer.

Consider the following integers (which depend on p):

$$\mu(p) = \frac{p^2}{4} + 2p + 2 = \frac{p}{2}\left(\frac{p}{2} + 4\right) + 2;$$
$$\gamma(p) = 2\mu(p) - \left(\frac{p}{2} + 4\right).$$

Definition

The numerical semigroup $\langle \mu, \gamma, \gamma + 1 \rangle_{p\mu}$, which contains $\{\mu, \gamma, \gamma + 1\}$ and all the integers greater than or equal to $p\mu$, is denoted by $S(p)$.

After some technical results one gets:

Corollary

The conductor of $S(p)$ is $p\mu$.

After some more technical results one gets:

Corollary

$|L| = \frac{p^3}{24} + \frac{3}{8}p^2 + \frac{13}{12}p.$

And also:

Corollary

$|D_p| = \left(\frac{p^2}{4} + \frac{5}{2}p + 6\right)/2.$

Now we can state the **main result**:

Theorem

$$\mathrm{E}(S(p)) = \frac{p}{4}\left(1 - \frac{p}{2}\right)$$

Corollary

There exist numerical semigroups with arbitrarily large negative Eliahou number.

The $S(p)$ satisfy Wilf's conjecture

From the previously stated results one gets:

Corollary

$|\mathrm{P}(S)| = \frac{p^2}{8} + \frac{3}{4}p + 2.$

And gets that the Wilf's conjecture holds for all numerical semigroups $S(p)$.

Proposition

Let p be an even positive integer. Then

$$\mathrm{W}(S(p)) = \frac{p^5}{192} + \frac{5p^4}{64} + \frac{p^3}{4} - \frac{7p^2}{16} + \frac{p}{6} > 0.$$

In particular, $\mathrm{W}(S(p)) > 0.$

Remark

When p grows, $\mathrm{E}(S(p))$ *becomes large negative, while* $\mathrm{W}(S(p))$ *becomes large positive.*

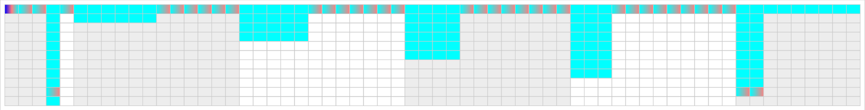
Defining the semigroup $S(p, \tau)$

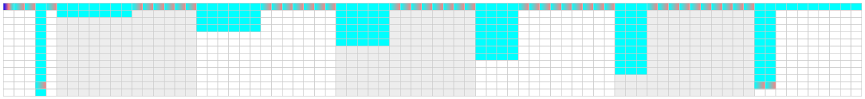
A visual way to look to $S(p, \tau)$

Let τ be a non-negative integer τ.

For each of the figures representing one $S(p)$ do the following slight modifications:

- add τ columns to the right of each of the B_i blocks, for $1 \leq i \leq \frac{p}{2} - 1$;
- concerning the block $B_{\frac{p}{2}}$, instead of adding columns to the right, τ columns are put at the left of the table. Block B is therefore split into two parts.
- Block C now consists of the columns $\tau + 1$ and $\tau + 2$.





It is straightforward to observe that the former images represent the semigroups $S(p,\tau)$ defined below.

Consider the integers:

$$m = \mu + \tau\tfrac{p}{2} = \tfrac{p}{2}\left(\tfrac{p}{2} + 4 + \tau\right) + 2 = \tfrac{p^2}{4} + 2p + 2 + \tau\tfrac{p}{2};$$
$$g = \gamma + \tau(p-1) = 2m - \left(\tfrac{p}{2} + 4 + \tau\right) = 2m - (\tau+1) - \left(\tfrac{p}{2} + 2 + 1\right);$$
$$c = p\mu + \tau\left(\tfrac{p}{2} - 1\right) = pm - \tau.$$

Definition

The numerical semigroup $\langle m, g, g+1\rangle_c$ is denoted by $S(p,\tau)$.

Remark

1. $|\mathrm{L}(S(p))| = |\mathrm{L}(S(p,\tau))|$;
2. $|D_p(S(p))| = |D_p(S(p,\tau))|$;

Theorem

Let p be an even positive integer and let τ be a non negative integer. Then $\mathrm{E}(S(p,\tau)) = \frac{p}{4}\left(1 - \frac{p}{2}\right) + \tau$.

Corollary

Every integer is the Eliahou number of some numerical semigroup of the form $S(p,\tau)$.

Remark

$|\mathrm{P}(S(p,\tau))| = |\mathrm{P}(S(p))| + \tau\frac{p}{2}$; $\mathrm{c}(S(p,\tau)) = \mathrm{c}(S(p)) + p\frac{p}{2}\tau - \tau$.

Proposition

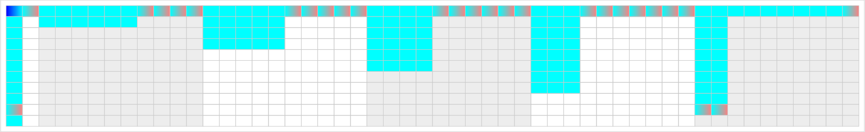
Let p be an even positive integer and let τ be a non negative integer. Then $\mathrm{W}(S(p,\tau)) > 0$.

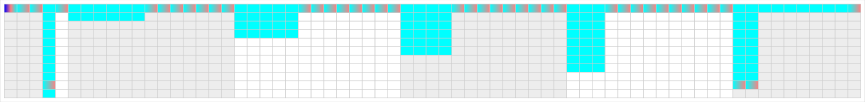
Defining the semigroups $S^{(i)}(p, \tau)$

A visual way to look to $S^{(i)}(p, \tau)$

Consider the image obtained by adding one column to the right of each of the B_i blocks...

In the case of the block $B_{\frac{p}{2}}$ we add one column to its right part.





The definition of $S^{(i)}(p,\tau)$

For p and τ as above, we define a sequence of semigroups $S^{(i)}(p,\tau)$, $i \geq 0$ as follows. We take $S^{(0)}(p,\tau) = S(p,\tau)$ and give the following recursive definition for the numbers involved.

$$m^{(i+1)} = m^{(i)} + \tfrac{p}{2};$$
$$g^{(i+1)} = g^{(i)} + p - 1;$$
$$c^{(i+1)} = pm^{(i)} + \tfrac{p^2}{2} - \tau = c^{(i)} + \tfrac{p^2}{2}.$$

Definition

Now, for $i \in \mathbb{Z}$, we define the semigroup $S^{(i)}(p,\tau)$ as the semigroup $\langle m^{(i)}, g^{(i)}, g^{(i)} + 1 \rangle_{c^{(i)}}$.

Remark

The sequence $S^{(i)}(p,\tau)$ is infinite.

One can easily observe that the constants involved in the Eliahou number do not change when we go from i to $i+1$... Thus:

Theorem

Let p be an even positive integer and let τ be a non negative integer. For any positive integer i, $\mathrm{E}(S(p,\tau)) = \mathrm{E}(S^{(i)}(p,\tau))$.

The $S^{(i)}(p,\tau)$ satisfy Wilf's conjecture:

Theorem

Let p be an even positive integer and let τ be a non negative integer. Then $\mathrm{W}(S^{(i)}(p,\tau)) > 0$, for every positive integer i.

The proof uses:

Lemma

$\mathrm{W}(S^{(i+1)}(p,\tau)) = \mathrm{W}(S^{(i)}(p,\tau)) + \frac{p^2(p^2+9p+2)}{48}$

Corollary

Given integers n and N, there are infinitely many numerical semigroups S such that $\mathrm{E}(S) = n$ and $\mathrm{W}(S) > N$.

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Grazie