Cyclotomic Numerical Semigroups I

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Preliminaries

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$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

from where, by Möbius inversion,

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)},$$

where μ denotes the Möbius' function.

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  In general, given a cyclotomic polynomial, or a product of cyclotomic polynomials, it is hard to say something about the coefficients.
  But: If such a polynomial were of the form $P_S(x)$ for some numerical semigroup S, then its non-zero coefficients would alternate between 1 and -1.

Cyclotomic Numerical Semigroups

Definition

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Lemma (Kronecker, 1857)

If f is a Kronecker polynomial with $f(0) \neq 0$, then all roots of f are on the unit circle and f factorizes as a product of cyclotomic polynomials.

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We say that a numerical semigroup S is cyclotomic of depth d and height h if $P_S(x) | (x^d - 1)^h$, where both d and h are chosen minimally, that is, $P_S(x)$ does not divide $(x^n - 1)^{h-1}$ for any n and it does not divide $(x^{d_1} - 1)^h$ for any $d_1 < d$.

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Lemma

Let S be a cyclotomic numerical semigroup. If

$$P_S(x) = \prod_{i=1}^{n} \Phi_{d_i}(x)^{e_i},$$

then S is of depth $d = \operatorname{lcm}(d_1, \ldots, d_n)$ and height $h = \max\{e_1, \ldots, e_n\}$.

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(Recall that S is symmetric if $S \cup (F(S) - S) = \mathbb{Z}$. This does not involve the roots of P_S.)

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Classify the cyclotomic numerical semigroups with prescribed depths and heights.

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Proof.

Use that S symmetric $\Leftrightarrow$ P_S selfreciprocal and the Lemma. □

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"$\Leftarrow$" Clear for $e(S) = 2$. If S is symmetric with $e(S) = 3$, then $S = \langle am_1, am_2, bm_1 + cm_2 \rangle$ with $a, b, c, m_1, m_2 \in \mathbb{N}$ such that $m_1, m_2, a, b + c \geq 2$ and $\gcd(m_1, m_2) = \gcd(a, bm_1 + cm_2) = 1$.

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The semigroup polynomial can then be easily computed:

$$P_S(x) = \frac{(1-x)(1-x^{am_1 m_2})(1-x^{a(bm_1+cm_2)})}{(1-x^{bm_1+cm_2})(1-x^{am_1})(1-x^{am_2})}.$$

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Two symmetric numerical semigroups with $F(S) = 11$ that are not cyclotomic:

$$S = \langle 5, 7, 8, 9 \rangle \text{ and } S = \langle 6, 7, 8, 9, 10 \rangle.$$

Symmetric Numerical Semigroups

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We suspect the following two families of symmetric numerical semigroups are not cyclotomic for $e \geq 4$. Using GAP, we verified this hypothesis up to multiplicity 30.

Example

$S = \langle m, m+1, qm+2q+2, \ldots, qm+(m-1) \rangle$, where m and q are positive integers such that $m \geq 2q+3$.

Example

$S = \langle m, m+1, (q+1)m+q+2, \ldots, (q+1)m+m-q-2 \rangle$, where m and q are non-negative integers such that $m \geq 2q+4$.

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By a Theorem of Assi et al. (2015) we then obtain

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and

$$P_S(x) = (1 - x)\prod_{i=1}^{t-1}(1 - x^{g_i}) \prod_{i=1}^{t}(1 - x^{a_i})^{-1}.$$

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If $e(S) \leq 3$ *then* S *cyclotomic* $\Leftrightarrow$ S *complete intersection* $\Leftrightarrow$ S *symmetric.*

In general, we have the inclusions

$$\{\text{complete intersection}\} \subseteq \{\text{cyclotomic}\} \subsetneq \{\text{symmetric}\}.$$

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$$\mathcal{K}(x) = H_S(x) \prod_{i=1}^{e} (1 - x^{n_i})$$

is a polynomial whose only non-zero terms are those of degree $n \in S$ such that the Euler characteristic of the shaded set of n, i.e. $\Delta_n = \left\{ L \subset \{n_1, \ldots, n_e\} : n - \sum_{s \in L} s \in S \right\}$, is not zero, that is, $\chi_S(n) = \sum_{L \in \Delta_n} (-1)^{\#L} \neq 0$.

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If S is cyclotomic, does $\mathcal{K}(x)$ factorize as $\prod_{b \in \mathrm{Betti}(S)} (1 - x^b)^{m_b}$?

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One might wonder whether an expression like

$$P_S(x) = (1-x)\prod_{i=1}^{t-1}(1-x^{g_i})\prod_{i=1}^{t}(1-x^{a_i})^{-1}$$

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is unique. In fact, more is true.

Lemma

If f is a polynomial with integer coefficients such that $f(0)=1$, then there exist unique $\epsilon_j \in \mathbb{Z}$ such that, for $|x|$ small enough,

$$f(x) = \prod_{j=1}^{\infty}(1-x^j)^{\epsilon_j}.$$

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As a consequence, given a numerical semigroup S, there are unique integers $\epsilon_1, \epsilon_2, \ldots$ such that

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Problem

Relate the properties of S to its cyclotomic exponent sequence.

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Under certain assumptions, if $S = \langle n_1, \ldots, n_e \rangle$ is cyclotomic and minimally generated, then there exist $k \in \mathbb{N}$, $1 < \delta_1 < \delta_2 < \cdots < \delta_k$ and $\epsilon_i \geq 1$, $i = 1, \ldots, k$ such that

$$H_S(x) = \frac{(1 - x^{\delta_1})^{\epsilon_1} \cdots (1 - x^{\delta_k})^{\epsilon_k}}{(1 - x^{n_1}) \cdots (1 - x^{n_e})}.$$

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Then $\delta_i \in S$ *for* $1 \leq i \leq k$ *and* $\delta_1 = \min\{s : s \in \text{Betti}(S)\}$.

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Proof.

Let $d(s)$ be the denumerant of $s \in S$.

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Then $\delta_i \in S$ *for* $1 \leq i \leq k$ *and* $\delta_1 = \min \{s : s \in \mathrm{Betti}(S)\}$.

Proof.

Let $d(s)$ be the denumerant of $s \in S$. Rewriting the above as

$$\sum_{s \in S} x^s = (1 - \epsilon_1 x^{\delta_1} + \cdots) \sum_{s \in S} d(s) x^s = \sum_{\substack{s \in S \\ s < \delta_1}} d(s) x^s + (d(\delta_1) - \epsilon_1) x^{\delta_1} + \cdots ,$$

it follows that δ_1 is the first $s \in S$ with $d(s) \geq 2$,

Attempts at Proving the Conjecture

As a first step confirming our hypothesis, we have the following.

Lemma

Suppose $S = \langle n_1, \ldots, n_e \rangle$ is cyclotomic, minimally generated and that

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it follows that δ_1 is the first $s \in S$ with $d(s) \ge 2$, hence the claim. □

Further Examples

Further Examples

Example (Free semigroups)

Let $S = \langle n_1, \ldots, n_t \rangle$. We say that S is free if either $S = \mathbb{N}$ or it is the gluing of the free semigroup $\langle n_1, \ldots, n_{t-1} \rangle$ and $\langle n_t \rangle$ (the order is important).

Let $n \geq 2$ and $(a_1, a_2, \ldots, a_n)$ be a sequence of coprime positive integers. For every $k = 1, \ldots, n$, let $d_k = \gcd(a_1, \ldots, a_k)$. For $k = 2, \ldots, n$, let $c_k = d_{k-1}/d_k$. Let S_k be the semigroup generated by $\{a_1, \ldots, a_k\}$. We say that the sequence $(a_1, a_2, \ldots, a_n)$ is smooth if $c_k a_k \in S_{k-1}$ for every $k = 2, \ldots, n$.

S is free iff S is generated by a smooth sequence.

If $S = \langle a_1, a_2, \ldots, a_n \rangle$ then, according to Leher's Ph.D. Thesis (2007),

$$P_S(x) = (1 - x) \prod_{i=2}^{n} (1 - x^{c_i a_i}) \prod_{i=1}^{n} (1 - x^{a_i})^{-1}.$$

Further Examples

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Example (Binomial semigroups)

Consider $B_n(a,b) = \langle a^n, ba^{n-1}, \ldots, ab^{n-1}, b^n \rangle$, where $a, b > 1$ are coprime. Putting $a_k = a^{n-k}b^k$ for $k = 0, \ldots, n$, the sequence $(a_0, \ldots, a_n)$ is smooth. We have

$$P_{B_n(a,b)}(x) = (1-x)\prod_{k=1}^{n}(1 - x^{a^{n+1-k}b^k})\prod_{k=0}^{n}(1 - x^{a^{n-k}b^k})^{-1}.$$

In particular, if p, q are distinct primes, we compute

$$P_{B_n(p,q)}(x) = \prod_{l=2}^{n+1}\prod_{\substack{i+j=l \\ 1 \le i,j \le l}} \Phi_{p^i q^j},$$

so that $B_n(p,q)$ is of depth $d = p^n q^n$ and height $h = 1$.

Semigroup Polynomial Divisors of $x^n - 1$

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Theorem

Let p, q, r be distinct primes. If S is cyclotomic of depth $d = pqr$ and height $h = 1$, then $S = \langle pq, r \rangle$ or a cyclic permutation.

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For $d = p^n q^n$, $n \geq 2$ and $h = 1$ we obtain $S = \langle p^n, q^n\rangle$ and the binomial semigroup $B_n(p, q) = \langle p^n, p^{n-1}q, \dots, pq^{n-1}, q^n\rangle$.

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We do not know whether these are all...

Polynomially Related Numerical Semigroups

Polynomially Related Numerical Semigroups

Definition

We say that the numerical semigroup S is polynomially related to the numerical semigroup T, and denote this by $S \leq_P T$, if there exist $f(x) \in \mathbb{Z}[x]$ and an integer $w \geq 1$ such that

$$H_S(x^w)f(x) = H_T(x),$$

or equivalently, $P_S(x^w)f(x) = P_T(x)(1 + x + \cdots + x^{w-1})$.

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Example

a) $\langle p^a, q^b \rangle \leq_P \langle p^m, q^n \rangle$ if $1 \leq a \leq m$ and $1 \leq b \leq n$.

b) $\langle p^a, q^b \rangle \leq_P B_n(p, q)$ if $a, b \geq 1$ and $2 \leq a + b \leq n + 1$.

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b) $\langle p^a, q^b \rangle \leq_P B_n(p, q)$ if $a, b \geq 1$ and $2 \leq a + b \leq n + 1$.

Problem

Find necessary and sufficient conditions such that $S \leq_P T$.

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In proving the following, we make repeated use of the fact that $P_S(1) = 1$ and $P'_S(1) = g(S)$.

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In proving the following, we make repeated use of the fact that $P_S(1) = 1$ and $P'_S(1) = g(S)$.

Lemma

Suppose that $H_S(x^w)f(x) = H_T(x)$ holds with S, T numerical semigroups. Then

a) $f(0) = 1$.

b) $f(1) = w$.

c) $f'(1) = w(g(T) - wg(S) + (w-1)/2)$.

d) $F(T) = wF(S) + \deg f$.

e) *If w is even, then $f(-1) = 0$.*

f) *If w is odd, then $f(-1) = P_T(-1)/P_S(-1)$.*

g) *If T is cyclotomic, then so is S.*

h) *If S is cyclotomic, then T is cyclotomic iff f is Kronecker.*

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Theorem

Let $p \neq q$ be primes and m, n positive integers. The quotient

$$Q(x) = P_{\langle p^m, q^n \rangle}(x) / \Phi_{p^m q^n}(x)$$

is in $\mathbb{Z}[x]$, is monic and has constant coefficient 1. Its non-zero coefficients alternate between 1 and -1.

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is in $\mathbb{Z}[x]$, is monic and has constant coefficient 1. Its non-zero coefficients alternate between 1 and -1.

In fact, a more general result holds.

Theorem

Suppose that S and T are numerical semigroups with $H_S(x^w) f(x) = H_T(x)$ for some $w \geq 1$ and $f \in \mathbb{N}[x]$. Put $Q(x) = P_T(x)/P_S(x^w)$. Then $Q(0) = 1$ and $Q(x)$ is a monic polynomial having non-zero coefficients that alternate between 1 and -1.

Thank you for attention!

Stay tuned for part II ;)