# On intersections of complete intersection ideals

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# Overview

# Introduction

Let $K$ be a field and $S = K[x_1, \dots x_r]$ be the polynomial ring in the variables $x_1, \dots, x_r$.

An ideal $I \subset S$ is called a complete intersection (CI for short) if it is minimally generated by $\mathtt{height}(I)$ elements.

This is a strong condition which is rarely preserved by taking intersections of such ideals.

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For an affine semigroup $H \subset \mathbb{N}$ the semigroup ring $K[H]$ is the subalgebra in $K[t]$ generated by the monomials $t^h$, $h \in H$. If $a_1, \dots, a_r$ generate $H$ minimally, we define the toric ideal $I_H$ as the kernel of the $K$-algebra map $\varphi : S \to K[H]$ letting $\varphi(x_i) = t^{a_i}$.

Consider the list of integers

$$\mathbf{a} = a_1 < a_2 < \cdots < a_r.$$

We denote $I(\mathbf{a})$ the kernel of the $K$-algebra map $\phi : S \to K[\langle \mathbf{a} \rangle]$ letting $\phi(x_i) = t^{a_i}$, where we let $\langle \mathbf{a} \rangle$ be the semigroup generated by $a_1, \ldots, a_r$. If they generate $\langle \mathbf{a} \rangle$ minimally, we call $I(\mathbf{a})$ the toric ideal of $\langle \mathbf{a} \rangle$.

If $k$ is any integer we let $\mathbf{a} + k = (a_1 + k, \ldots, a_r + k)$ and we denote $I(\mathbf{a} + k)$ the toric ideal of $\langle \mathbf{a} + k \rangle$.

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The study of properties of the family of ideals $\{I(\mathbf{a} + k)\}_{k \geq 0}$ is a recent topic of interest. First notice that for $k \gg 0$, $a_1 + k, \ldots, a_r + k$ generate the semigroup $\langle \mathbf{a} + k \rangle$ minimally.

# Introduction

A result of Delorme [2] characterizes semigroups $H$ such that $I_H$ is CI. This is an arithmetic property of the semigroup, it does not depend on the field $K$. A semigroup $H$ is called a complete intersection if $K[H]$ has this property.

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$$\text{for all } k \gg 0 \qquad I(\mathbf{a}+k) \text{ is CI} \iff I(\mathbf{a}+k+(a_r-a_1)) \text{ is CI.}$$

This was a particular case of a conjecture of Herzog and Srinivasan [4], proved in full generality by Vu in [8]:

$$\text{for all } k \gg 0 \qquad \beta_i^S(I(\mathbf{a}+k)) = \beta_i^S(I(\mathbf{a}+k+(a_r-a_1))), \text{ for all } i.$$

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We use Gröbner bases techniques to derive new information about intersections of such CI ideals. As a main result we prove that for a fixed sequence $\mathbf{a}$ and $k \gg 0$, intersections of CI-ideals $I(\mathbf{a}+k)$ produce another CI ideal.

# Main results

Firstly, we fix some notations.
For $0 \neq f$ and $I$ ideal in $S$ we let

$$\begin{aligned} \mathrm{in}_<(f) &:= \text{leading term with respect to revlex} \\ f^* &:= \text{the (nonzero) homogeneous component of least degree} \\ I^* &:= (f^* : f \in I, f \neq 0) \quad \text{the ideal of initial forms of } I \\ \bar{I} &:= \text{the homogenization of } I \text{ w.r.t. a new variable } x_0. \end{aligned}$$

$I^*$ appears naturally as the defining ideal of the associated graded ring of $S/I$ with respect to $\mathfrak{m} = (x_1, \dots, x_r)$, i.e.

$$S/I^* \cong \mathrm{gr}_{\mathfrak{m}}(S/I).$$

The polynomials $f_1, \dots, f_s$ are called a standard basis for $I$ if $I^* = (f_1^*, \dots, f_s^*)$.

# Main results

Here is a first result, inspired by the work of Jayanthan and Srinivasan ([5]).

**Theorem 1 (C., Stamate, 2016)**

Consider the sequence $\mathbf{a} = a_1 < \cdots < a_r$ and let $k \geq (a_r - a_1)^2 - a_1$ such that $\mathbf{a} + k$ is CI. Then $I(\mathbf{a} + k)$ is minimally generated by a reduced Gröbner basis computed with respect to revlex.

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As a direct consequence, we get the following Corollary.

## Corollary 1

If $k \geq (a_r - a_1)^2 - a_1$ and $I(\mathbf{a} + k)$ is CI, then $\bar{I}(\mathbf{a} + k)$ and $I^*(\mathbf{a} + k)$ are CI, too, and they are minimally generated by their respective reduced Gröbner basis with respect to revlex.

In the proof we also used the CI-splits introduced by Delorme.

# Main results

For a sequence of nonnegative integers $\mathbf{a} = a_1, \ldots, a_r$ we let

$$J(\mathbf{a}) = (f \in I(\mathbf{a}) : f \text{ is homogeneous}) \subseteq S.$$

It is easy to see that

$$J(\mathbf{a}) = J(\mathbf{a} + k) \text{ for all } k \geq 0$$

Also, $J(\mathbf{a})$ is the toric ideal of the semigroup $\langle (a_1, 1), \ldots, (a_r, 1) \rangle \subset \mathbb{Z}^2$, hence $J(\mathbf{a})$ is a prime ideal in $S$ of height $r - 2$.

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**Corollary 2**

With notation as above, if $\mathbf{a} + k$ is CI for some $k \geq (a_r - a_1)^2 - a_1$, then $J(\mathbf{a})$ is CI. Moreover, $J(\mathbf{a})$ is minimally generated by a reduced Gröbner basis with respect to revlex.

# Main results

Let $\mathbf{a} = a_1 < \cdots < a_r$ be a sequence of nonnegative integers and $\mathcal{A} \subset \mathbb{N}$. We introduce:

$$\begin{aligned}
\mathcal{I}_{\mathcal{A}}(\mathbf{a}) &= \bigcap_{k\in\mathcal{A}} I(\mathbf{a}+k), \\
\mathcal{J}_{\mathcal{A}}(\mathbf{a}) &= \bigcap_{k\in\mathcal{A}} I^*(\mathbf{a}+k), \\
\mathcal{H}_{\mathcal{A}}(\mathbf{a}) &= \bigcap_{k\in\mathcal{A}} \bar{I}(\mathbf{a}+k).
\end{aligned}$$

Our aim is to understand these ideals.

# Main results

The next result shows that when we intersect infinitely many toric ideals (or the ideals of their initial forms) in the same shifted family, the result does not the depend on the family $\mathcal{A}$ of shifts.

## Proposition 1

Assume $\mathcal{A}$ is an infinite set of nonnegative integers. Then:

$$\begin{aligned} \mathcal{I}_{\mathcal{A}}(\mathbf{a}) &= \mathcal{J}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a}), \\ \mathcal{H}_{\mathcal{A}}(\mathbf{a}) &= J(\mathbf{a})S[x_0]. \end{aligned}$$

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We shall now state our main result.

## Theorem 2 (C., Stamate, 2016)

Let $\mathcal{A} \subseteq \mathbb{N}$ such that $\min \mathcal{A} \geq (a_r - a_1)^2 - a_1$. If $I(\mathbf{a}+k)$ is CI for all $k \in \mathcal{A}$, then $\mathcal{I}_{\mathcal{A}}(\mathbf{a})$, $\mathcal{J}_{\mathcal{A}}(\mathbf{a})$, $\mathcal{H}_{\mathcal{A}}(\mathbf{a})$ and $\mathcal{I}_{\mathcal{A}}(\mathbf{a})^*$ are CI, as well.

**The idea of the proof**. Since $k \geq (a_r - a_1)^2 - a_1$, we are able to prove that:

$$I(\mathbf{a}+k) = J(\mathbf{a}) + (f_{r-1,k}),$$

where $f_{r-1,k} = x_r^{(a_1+k)/d_k} - x_1^{(a_r+k)/d_k}$ and $d_k = \gcd(a_1+k, a_r+k)$.
Denote $f_{\mathcal{A}} = \operatorname{lcm}(f_{r-1,k} : k \in \mathcal{A})$.
We show that

$$\mathcal{I}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a}) + (f_{\mathcal{A}}).$$

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The statement about $\mathcal{H}_{\mathcal{A}}(\mathbf{a})$ is proven similarly.

For $k$ in our range, one obtains via Gröbner bases that $I(\mathbf{a}+k)$ is generated by a standard basis. Therefore $I^*(\mathbf{a}+k) = J(\mathbf{a}) + (f^*_{r-1,k}) = J(\mathbf{a}) + (x_r^{\beta_k})$, hence $\mathcal{J}_{\mathcal{A}}(\mathbf{a}) = I^*(\mathbf{a}+k_0)$ for some $k_0$ in $\mathcal{A}$.

# Examples

The statements of Theorem 2 hold for shifts $k \gg 0$. Even though $I(\mathbf{a} + k)$ may be CI for infinitely many $k$, when we intersect two CI ideals $I(\mathbf{a} + k_1)$ and $I(\mathbf{a} + k_2)$ for $k_1$ or $k_2$ not large enough, the result might not be again a CI.

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Let $\mathbf{a} = 0, 6, 15$. As noted in the paper of Stamate [7], for $k \geq 25$, $I(\mathbf{a}+k)$ is CI if and only if $k = 5\ell$. Still, the ideal $I(\mathbf{a}+k) \subset K[x,y,z]$ is a CI for $k = 8$ and $k = 10$ and a SINGULAR computation shows that:

$$\begin{aligned} I(8,14,24) \cap I(10,16,25) &= (z^2 - x^4y, x^7 - y^4) \cap (y^5 - x^3z^2, x^5 - z^2) \\ &= (y^5 - x^3z^2, x^9y - x^5z^2 - x^4yz^2 + z^4, x^{12} - x^5y^4 - x^7z^2 + y^4z^2) \end{aligned}$$

is not a CI.

# Examples

It is natural to ask if the CI property may be replaced by Gorenstein in Theorem 2. We give a negative answer, in the following Example:

Let $\mathbf{a} = 0, 1, 2, 3$. According to a result of Patil and Sengupta [6, Corollary 6.2], the ideal $I(\mathbf{a} + k) \subset S = K[x_1, x_2, x_3, x_4]$ is Gorenstein if and only if $k \equiv 2 \mod 3$. Set $I_k = I(\mathbf{a} + k)$.

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Assume $k = 3\ell + 2$. Then, we have:

$$I_k = (x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4, x_1^{\ell+1}x_2 - x_4^{\ell+1}, x_1^{\ell+2} - x_4^{\ell}x_3).$$

We also get

$$J(\mathbf{a}) = (x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4).$$

# Examples

Using any algorithm for computing the intersection we get:

$$
\begin{array}{rrrrr}
I_k \cap I_{k+3} = J(\mathbf{a})+( & x_1^{2\ell+4}x_3- & x_1^{\ell+2}x_2x_4^{\ell+1}- & x_1^{\ell+1}x_2x_4^{\ell+2}+ & x_4^{2\ell+3}, \\
 & x_1^{2\ell+4}x_2- & x_1^{\ell+3}x_4^{\ell+1}- & x_1^{\ell+2}x_4^{\ell+2}+ & x_3x_4^{2\ell+2}, \\
 & x_1^{2\ell+5}- & x_1^{\ell+3}x_3x_4^{\ell}- & x_1^{\ell+2}x_3x_4^{\ell+1}+ & x_2x_4^{2\ell+2}).
\end{array}
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As $x_1$ is regular on both $S/I_k$ and $S/I_{k+3}$, it is regular on $S/(I_k \cap I_{k+3})$, too. Using reduction modulo $x_1$ it is enough to show that the $(x_1, I_k \cap I_{k+3})$ is not a Gorenstein ideal.

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Letting $R = S/(x_1, I_k \cap I_{k+3})$ we notice that

$$(x_1, I_k \cap I_{k+3}) = (x_1, x_3^2 - x_2x_4, x_2x_3, x_2^2, x_4^{2\ell+3}, x_2x_4^{2\ell+2}, x_3x_4^{2\ell+2})$$

and the residue classes $u = \widehat{x_4^{2\ell+2}}$ and $v = \widehat{x_2x_4^{2\ell+1}}$ are linear independent in $Soc(R)$.

# Further developments

Several periodic features have been noticed for the Betti numbers of the toric ideal and other ideals attached to large enough shifts of a numerical semigroup. We summarize the most important ones below.
Let $\mathbf{a} = a_1 < \cdots < a_r$ be a sequence of integers. Then:

**Theorem**

For $k \gg 0$ and all $i$ one has

(i) (Vu, [8, Theorem 1.1]) $\beta_i(I(\mathbf{a}+k)) = \beta_i(I(\mathbf{a}+k+(a_r-a_1)))$,

(ii) (Herzog–Stamate, [4, Theorem 1.4]) $\beta_i(I(\mathbf{a}+k)) = \beta_i(I(\mathbf{a}+k)^*)$ ,

(iii) (Vu, [8, Theorem 5.7]) $\beta_i(I(\mathbf{a}+k)) = \beta_i(\bar{I}(\mathbf{a}+k))$.

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Numerical experiments with SINGULAR encourage us to believe that similar periodicities occur for the Betti numbers of intersections of these ideals, as well. If $\mathcal{A}$ is any set of integers and $k$ is any integer we denote $\mathcal{A}+k = \{a+k : a \in \mathcal{A}\}$.

# Further developments

We propose therefore the following Conjecture.

## Conjecture 1

With notation as above, for $k \gg 0$ and all $i$ one has

(i) $\beta_i(\mathcal{I}_{\mathcal{A}+k}(\mathbf{a})) = \beta_i(\mathcal{I}_{\mathcal{A}+k+(a_r-a_1)}(\mathbf{a}))$,

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As a consequence of Theorem 2, we get:

## Proposition 2

In any of the following situations, Conjecture 1 holds:

(i) $\mathcal{A}$ is infinite,

(ii) $I(\mathbf{a}+k)$ is CI for all $k \in \mathcal{A}$ and $\min \mathcal{A} \geq (a_r - a_1)^2 - a_1$.

# Further developments

Note that Conjecture 1(*iii*) can not be improved by adding the equality $\beta_i(\mathcal{I}_{\mathcal{A}}(\mathbf{a})) = \beta_i(\mathcal{J}_{\mathcal{A}}(\mathbf{a}))$, which is nevertheless true when $|\mathcal{A}| = 1$.
Indeed, continuing the previous example with $\mathbf{a} = 0, 1, 2, 3$ and $k = 3\ell + 2$ one has

$$I_k^* = J(\mathbf{a}) + (x_4^{\ell+1}, x_4^{\ell} x_3).$$

Clearly $I_k^* \supset I_{k+3}^*$, and using [4, Proposition 2.5] both are Gorenstein ideals because $I_k$ and $I_{k+3}$ are so. Thus $I_k^* \cap I_{k+3}^* = I_{k+3}^*$ is a Gorenstein ideal, and this shows that in general

$$\beta_i(I_k \cap I_{k+3}) \neq \beta_i(I_k^* \cap I_{k+3}^*).$$

# Further developments

In a new paper (work under progress; almost finished), we give a proof of the Conjecture 1($ii$) and we make some steps in order to prove Conjecture 1($i$). In order to prove Conjecture 1($ii$), we used the fact that the ideals $I^*(\mathbf{a}+k)$'s are generated by homogeneous binomials (which are those of $J(\mathbf{a})$) and monomials. Thus we can manage their intersection and we can also compare $I^*(\mathbf{a}+k)$ with $I^*(\mathbf{a}+k+a_r-a_1)$.

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On the other hand, Conjecture 1($i$) is more difficult, since the intersection of ideals $I(\mathbf{a}+k)$'s is much harder to express. One possible way to advance in this problem, is to use reduction modulo $x_1$, and we have some partial results in this direction.

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# Thank you!