Seeds of numerical semigroups

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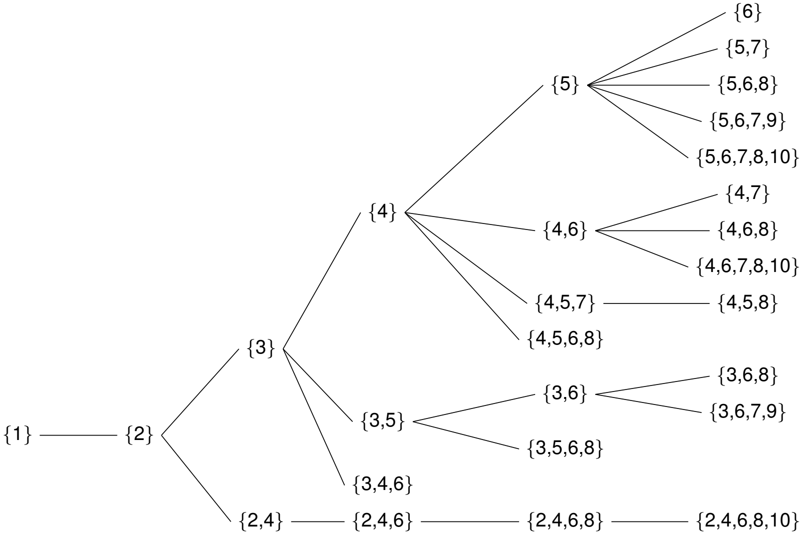
The tree of numerical semigroups



Semigroups are represented by the non-zero non-gaps up to the conductor

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They are in bijection with its immediate descendants in the semigroup tree.

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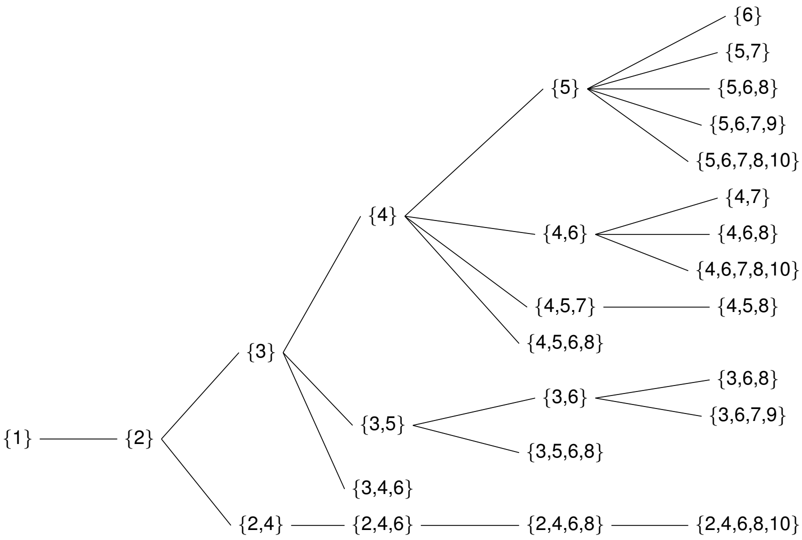

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The table of seeds of a semigroup

Lemma. Any order-i seed of Λ is at most $c + \lambda_{i+1} - \lambda_i - 1$.

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order 0	1	1	0	1	0	0	0	0	$\lambda_1 - \lambda_0 = 8$
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Suppose λ_s is an order-zero seed of Λ ($s \geqslant k$).

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$s=2$	$s=3$	$s=5$	$s=6$
$\tilde{\Lambda} = \{0, 5, 9, 10, \dots\}$	$\tilde{\Lambda} = \{0, 5, 8, 10, \dots\}$	$\tilde{\Lambda} = \{0, 5, 8, 9, 10, 12, \dots\}$	$\tilde{\Lambda} = \{0, 5, 8, 9, 10, 11, 13, \dots\}$

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Goal. Obtain the seeds of $\tilde{\Lambda}$ from those of Λ.

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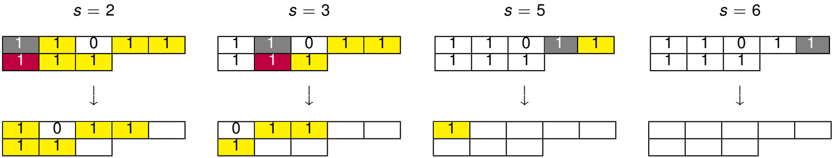

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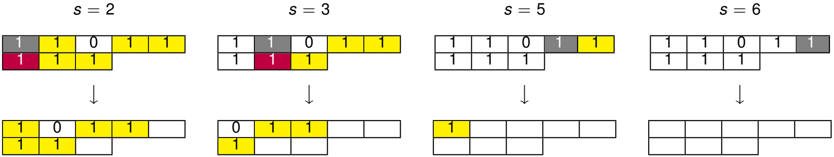

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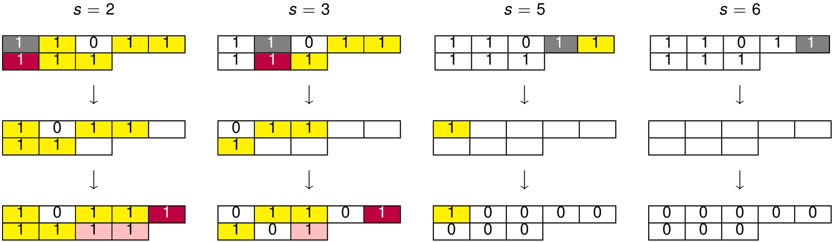

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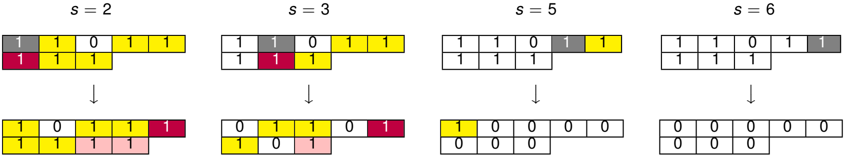
- If $i < s - 2$, then $\tilde{\Lambda}$ has no order-i seeds.
- If $i = s - 2$, then the only order-i seed of $\tilde{\Lambda}$ is $\lambda_s + 1$.
- If $i = s - 1$, then the only order-i seeds of $\tilde{\Lambda}$ are $\lambda_s + 1$ and $\lambda_s + 2$.



New-order seeds

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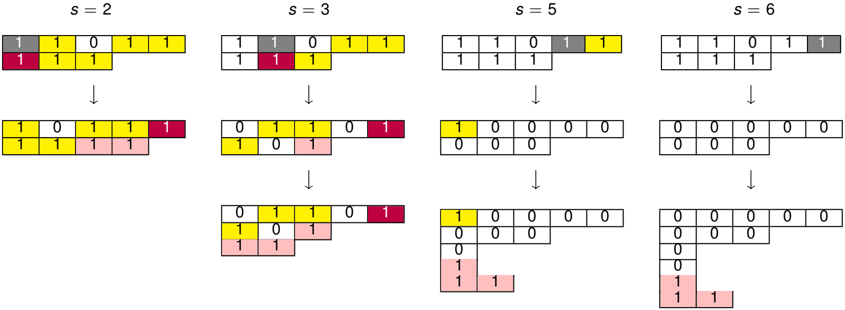

Example (A). $\Lambda = \{0, 8, 10, 11, \underbrace{\overbrace{\mathbf{14}}^{\lambda_4}}_{c}, \overbrace{\mathbf{15}}^{\lambda_5}, 16, \overbrace{\mathbf{17}}^{\lambda_7}, 18, \dots\}$

Example (A). $\Lambda = \{0, 8, 10, 11, \underbrace{\overbrace{\mathbf{14}}^{\lambda_4}}_{c}, \overbrace{\mathbf{15}}^{\lambda_5}, 16, \overbrace{\mathbf{17}}^{\lambda_7}, 18, \dots\}$

$s = 4$

1	1	0	1	0	0	0	0
0	1						
1							
1	1	1					

$s = 5$

1	1	0	1	0	0	0	0
0	1						
1							
1	1	1					

$s = 7$

1	1	0	1	0	0	0	0
0	1						
1							
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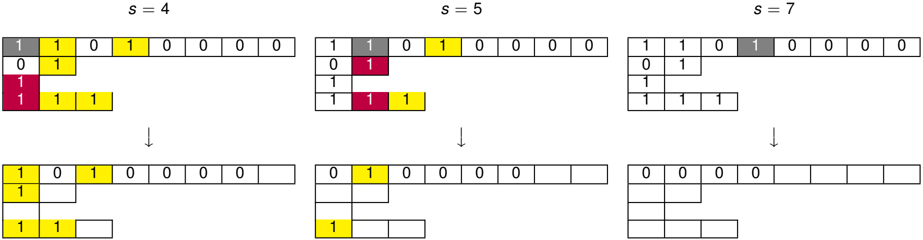

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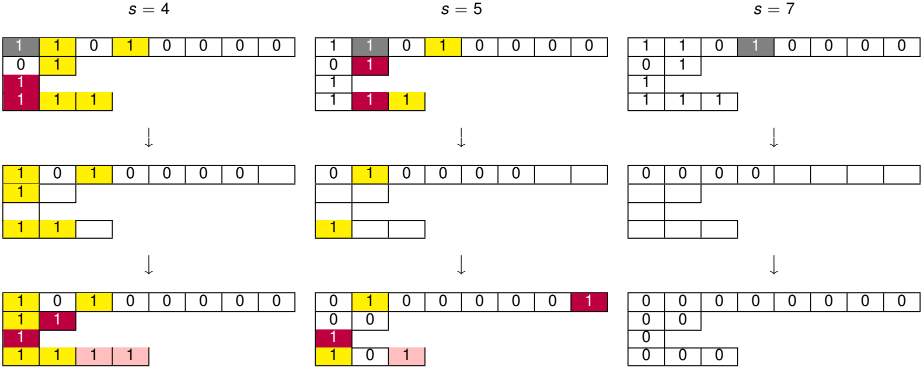

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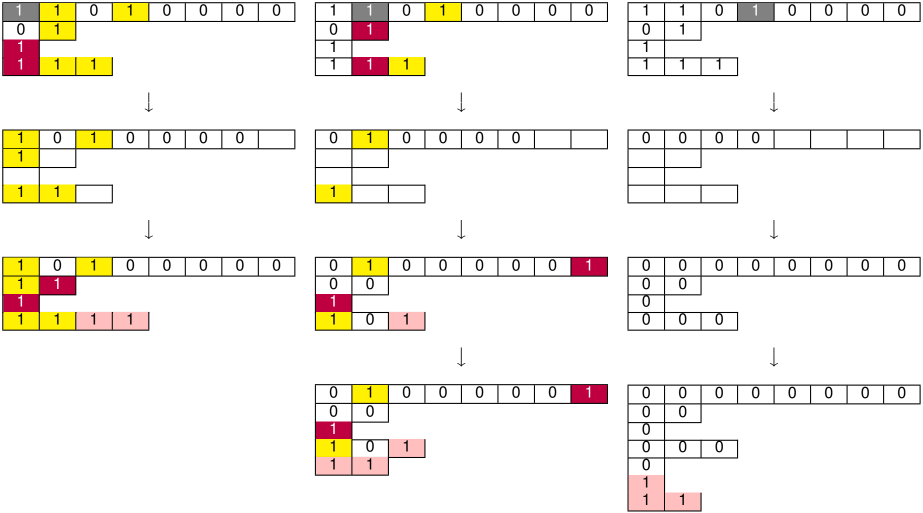

Strings G, S

The string $G = G(\Lambda)$ stores the gaps of Λ.

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0	1						
1							
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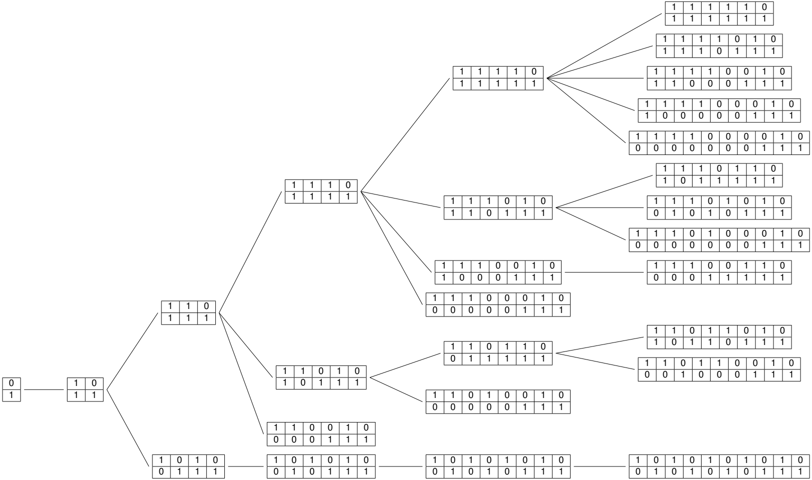
the strings G, S are

$G \rightarrow$	1	1	1	1	1	1	1	0	1	0	0	1	1	0
$S \rightarrow$	1	1	0	1	0	0	0	0	0	1	1	1	1	1

G, S in the tree of semigroups



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Update of S:

Let $\tilde{S} = \tilde{S}_0\, \tilde{S}_1 \cdots \tilde{S}_\ell \cdots$ with

$$\tilde{S}_\ell := \begin{cases} 0 & \text{if} \quad \ell = \lambda_i + j \quad \text{with} \quad 1 \leqslant i < k,\ 0 \leqslant j < \Delta, \\ S_\ell & \text{otherwise.} \end{cases}$$

Then,

$$S(\tilde{\Lambda}) = \tilde{S}_{\Delta+1}\, \tilde{S}_{\Delta+2} \cdots \tilde{S}_{c-1}\, \overbrace{0 \cdots 0\ 1}^{2\Delta}\ 1\ 1.$$

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It is useful to manipulate $G(\Lambda)$ and $S(\Lambda)$ as integers in binary form.

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Then,

$$\begin{aligned} G(\tilde{\Lambda}) &= G \mid (1 \gg \tilde{c} - 2) \\ S(\tilde{\Lambda}) &= (\tilde{S} \ll \Delta + 1) \mid (1\,1\,1 \gg c + \Delta - 2) \end{aligned}$$

Descending algorithm for G, S

Input: $c := c(\Lambda),\ \ G := G(\Lambda),\ \ S := S(\Lambda),\ \ \Delta$

Output: $c(\tilde{\Lambda}),\ \ G(\tilde{\Lambda}),\ \ S(\tilde{\Lambda})$

1. $\tilde{S} := S$
2. $\texttt{rake} := G$
3. **from** 1 **to** Δ **do**
4. $\quad \texttt{rake} := \texttt{rake} \gg 1$
5. $\quad \tilde{S} := \tilde{S}\ \&\ \texttt{rake}$
6. **return** $\tilde{c} := c + \Delta + 1,\ \ G \mid (1 \gg \tilde{c} - 2),\ \ (\tilde{S} \ll \Delta + 1) \mid (1\,1\,1 \gg \tilde{c} - 3)$

Comparing algorithms

Time in seconds to compute n_g:

	30	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
Apéry - DFS	13	24	39	67	114	193	327	554	933	1577	2657
Apéry - recursive	10	16	28	47	81	136	232	393	634	1071	1805
decomposition - DFS	10	16	27	46	79	131	222	373	626	1050	1762
single check - DFS	8	14	23	39	65	110	185	310	518	868	1448
decomposition - recursive	7	12	20	35	58	97	165	275	462	775	1297
single check - recursive	2	4	7	11	19	31	53	87	145	241	400
seeds - DFS	1	3	4	8	12	21	35	58	96	161	269
seeds - recursive	1	2	3	6	9	15	26	42	70	118	195