# Counting numerical semigroups of a given genus by using $\gamma$-hyperelliptic semigroups

**Matheus Bernardini**[1]
University of Campinas / Federal Institute of São Paulo

(Joint work in progress with Fernando Torres[1])

International Meeting on Numerical Semigroups with Applications
Levico Terme, July 5 2016

[1]Both authors are partially supported by CNPq

# Introduction

Let $S$ be a numerical semigroup.

- $G(S) := \mathbb{N}_0 \setminus S$ - set of gaps of $S$;
- $g(S) := \#G(S)$ - genus of $S$;
- $n_g := \#\{S : g(S) = g\}$.

## Examples

- $n_0 = 1 \ - \mathbb{N}_0$
- $n_1 = 1 \ - \mathbb{N}_0 \setminus \{1\}$
- $n_2 = 2 \ - \mathbb{N}_0 \setminus \{1, 2\}$ and $\mathbb{N}_0 \setminus \{1, 3\}$
- $n_3 = 4 \ - \mathbb{N}_0 \setminus \{1, 2, 3\}, \mathbb{N}_0 \setminus \{1, 2, 4\}, \mathbb{N}_0 \setminus \{1, 2, 5\}$ and $\mathbb{N}_0 \setminus \{1, 3, 5\}$

### Interest

Studying the behavior of $n_g$.

### Main Goal (but still not solved)

$$n_g \leq n_{g+1}, \text{ for all } g.$$

## Interest

Studying the behavior of $n_g$.

## Main Goal (but still not solved)

$$n_g \leq n_{g+1}, \text{ for all } g.$$

# A brief survey

## First Bound

If $g(S) = g$, then $2g + \mathbb{N}_0 \subset S$. Hence,

$$n_g \leq \binom{2g-1}{g}.$$

## M. Bras-Amorós and A. de Mier - 2007

$$n_g \leq C_g = \frac{1}{g+1}\binom{2g}{g}.$$

# A brief survey

## First Bound

If $g(S) = g$, then $2g + \mathbb{N}_0 \subset S$. Hence,

$$n_g \leq \binom{2g-1}{g}.$$

## M. Bras-Amorós and A. de Mier - 2007

$$n_g \leq C_g = \frac{1}{g+1}\binom{2g}{g}.$$

From now on, let $\varphi = \frac{1+\sqrt{5}}{2}$.

## M. Bras-Amorós - 2006/2008 (Conjecture)

1.
   - $\lim\limits_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi$;
   - $\lim\limits_{g\to\infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1$.
2. $n_g + n_{g+1} \leq n_{g+2}$, *for all* $g$.

## M. Bras-Amorós - 2009

*Let* $(F_n)_{n\geq 0} = (1,1,2,3,5,8,13,\dots)$ *be the Fibonacci sequence. Then*

$$2F_g \leq n_g \leq 1 + 3 \cdot 2^{g-3}, \forall g \geq 3.$$

## S. Elizalde - 2010

$$a_g \leq n_g \leq c_g, \forall g \geq 1$$

*where* $a_g$ *and* $c_g$ *are coefficients of some explicit generating functions.*

## M. Bras-Amorós - 2009

*Let* $(F_n)_{n\geq 0} = (1, 1, 2, 3, 5, 8, 13, \ldots)$ *be the Fibonacci sequence. Then*

$$2F_g \leq n_g \leq 1 + 3 \cdot 2^{g-3}, \forall g \geq 3.$$

## S. Elizalde - 2010

$$a_g \leq n_g \leq c_g, \forall g \geq 1$$

*where* $a_g$ *and* $c_g$ *are coefficients of some explicit generating functions.*

## A. Zhai - 2011/2013

1. $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi;$
2. $\lim_{g\to\infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1.$

## Remark

- Zhai's first item implies that $n_g < n_{g+1}$, for $g \gg 0$.
- Checking if $n_g \leq n_{g+1}$ for all $g$ is still an open problem (weaker conjecture).

## A. Zhai - 2011/2013

1. $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi$;
2. $\lim_{g\to\infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1$.

## Remark

- Zhai's first item implies that $n_g < n_{g+1}$, for $g \gg 0$.
- Checking if $n_g \leq n_{g+1}$ for all $g$ is still an open problem (weaker conjecture).

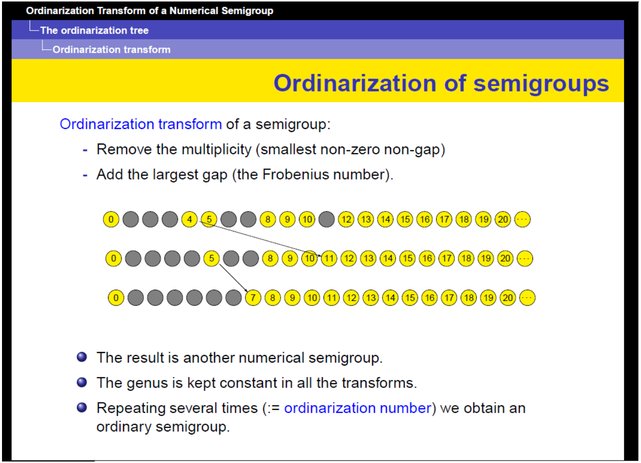
- $m(S) := \min\{s \in S : s \neq 0\}$ - multiplicity of $S$;
- $N(m, g) := \#\{S : g(S) = g \text{ and } m(S) = m\}$.

**N. Kaplan - 2012**

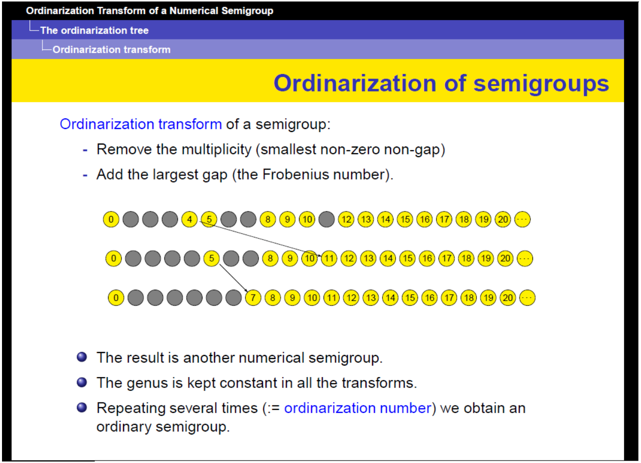
*If $2g < 3m$, then*

$$N(m, g) = N(m-1, g-1) + N(m-1, g-2).$$

| $g\backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | $N(g)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 2 | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 3 | | 1 | 2 | 1 | | | | | | | | | | | | | | | | | | | | | | | 4 |
| 4 | | 1 | 2 | 3 | 1 | | | | | | | | | | | | | | | | | | | | | | 7 |
| 5 | | 1 | 2 | 4 | 4 | 1 | | | | | | | | | | | | | | | | | | | | | 12 |
| 6 | | 1 | 3 | 6 | 7 | 5 | 1 | | | | | | | | | | | | | | | | | | | | 23 |
| 7 | | 1 | 3 | 7 | 10 | 11 | 6 | 1 | | | | | | | | | | | | | | | | | | | 39 |
| 8 | | 1 | 3 | 9 | 13 | 17 | 16 | 7 | 1 | | | | | | | | | | | | | | | | | | 67 |
| 9 | | 1 | 4 | 11 | 16 | 27 | 28 | 22 | 8 | 1 | | | | | | | | | | | | | | | | | 118 |
| 10 | | 1 | 4 | 13 | 22 | 37 | 44 | 44 | 29 | 9 | 1 | | | | | | | | | | | | | | | | 204 |
| 11 | | 1 | 4 | 15 | 24 | 49 | 64 | 72 | 66 | 37 | 10 | 1 | | | | | | | | | | | | | | | 343 |
| 12 | | 1 | 5 | 18 | 32 | 66 | 85 | 116 | 116 | 95 | 46 | 11 | 1 | | | | | | | | | | | | | | 592 |
| 13 | | 1 | 5 | 20 | 35 | 85 | 112 | 172 | 188 | 182 | 132 | 56 | 12 | 1 | | | | | | | | | | | | | 1001 |
| 14 | | 1 | 5 | 23 | 43 | 106 | 148 | 239 | 288 | 304 | 277 | 178 | 67 | 13 | 1 | | | | | | | | | | | | 1693 |
| 15 | | 1 | 6 | 26 | 51 | 133 | 191 | 325 | 409 | 492 | 486 | 409 | 234 | 79 | 14 | 1 | | | | | | | | | | | 2857 |
| 16 | | 1 | 6 | 29 | 61 | 163 | 237 | 441 | 559 | 754 | 796 | 763 | 587 | 301 | 92 | 15 | 1 | | | | | | | | | | 4806 |
| 17 | | 1 | 6 | 32 | 68 | 196 | 301 | 573 | 750 | 1094 | 1246 | 1282 | 1172 | 821 | 380 | 106 | 16 | 1 | | | | | | | | | 8045 |
| 18 | | 1 | 7 | 36 | 80 | 236 | 369 | 737 | 1015 | 1534 | 1841 | 2074 | 2045 | 1759 | 1122 | 472 | 121 | 17 | 1 | | | | | | | | 13467 |
| 19 | | 1 | 7 | 39 | 89 | 282 | 444 | 945 | 1334 | 2106 | 2601 | 3227 | 3356 | 3217 | 2580 | 1502 | 578 | 137 | 18 | 1 | | | | | | | 22464 |
| 20 | | 1 | 7 | 43 | 104 | 330 | 541 | 1193 | 1737 | 2840 | 3561 | 4812 | 5301 | 5401 | 4976 | 3702 | 1974 | 699 | 154 | 19 | 1 | | | | | | 37396 |
| 21 | | 1 | 8 | 47 | 115 | 390 | 658 | 1490 | 2231 | 3793 | 4822 | 6939 | 8020 | 8721 | 8618 | 7556 | 5204 | 2552 | 836 | 172 | 20 | 1 | | | | | 62194 |
| 22 | | 1 | 8 | 51 | 133 | 456 | 784 | 1847 | 2851 | 4967 | 6490 | 9752 | 11657 | 13643 | 14122 | 13594 | 11258 | 7178 | 3251 | 990 | 191 | 21 | 1 | | | | 103246 |
| 23 | | 1 | 8 | 55 | 143 | 525 | 940 | 2262 | 3582 | 6426 | 8638 | 13439 | 16375 | 20640 | 22364 | 22740 | 21150 | 16462 | 9730 | 4087 | 1162 | 211 | 22 | 1 | | | 170963 |
| 24 | | 1 | 9 | 60 | 164 | 608 | 1114 | 2766 | 4475 | 8252 | 11355 | 18309 | 22518 | 30282 | 34252 | 36614 | 36334 | 32408 | 23640 | 12981 | 5077 | 1353 | 232 | 23 | 1 | | 282828 |
| 25 | | 1 | 9 | 64 | 181 | 697 | 1307 | 3353 | 5512 | 10519 | 14756 | 24663 | 30588 | 43234 | 50772 | 57340 | 59354 | 57484 | 48870 | 33370 | 17068 | 6239 | 1564 | 254 | 24 | 1 | 467224 |




- Part of M. Bras-Amorós' presentation in last IMNS;
- $r(S)$ - ordinarization number of $S$;
- $n_{g,r} := \#\{S : g(S) = g \text{ and } r(S) = r\}$.




- Part of M. Bras-Amorós' presentation in last IMNS;
- $r(S)$ - ordinarization number of $S$;
- $n_{g,r} := \#\{S : g(S) = g \text{ and } r(S) = r\}$.

### M. Bras-Amorós - 2012

*If* $r > max\{\frac{g}{3}+1, \left\lfloor \frac{g+1}{2} \right\rfloor - 14\}$, *then* $n_{g,r} \leq n_{g+1,r}$.

### M. Bras-Amorós - 2012 (Conjecture)

*If* $r > \frac{g}{3}$, *then* $n_{g,r} \leq n_{g+1,r}$.

**M. Bras-Amorós - 2012**

*If* $r > max\{\frac{g}{3} + 1, \left\lfloor \frac{g+1}{2} \right\rfloor - 14\}$, *then* $n_{g,r} \leq n_{g+1,r}$.

**M. Bras-Amorós - 2012 (Conjecture)**

*If* $r > \frac{g}{3}$, *then* $n_{g,r} \leq n_{g+1,r}$.

# Our approach

- $\gamma(S)$: number of even gaps of $S$ - $\#[G(S) \cap 2\mathbb{Z}]$;
- $\gamma$-hyperelliptic semigroup: numerical semigroup with $\gamma$ even gaps;
- $N_\gamma(g) := \#\{S : g(S) = g \text{ and } \gamma(S) = \gamma\}$.

$$n_g = \sum_{\gamma=0}^{g} N_\gamma(g)$$

# Our approach

- $\gamma(S)$: number of even gaps of $S$ - $\#[G(S) \cap 2\mathbb{Z}]$;
- $\gamma$-hyperelliptic semigroup: numerical semigroup with $\gamma$ even gaps;
- $N_\gamma(g) := \#\{S : g(S) = g \text{ and } \gamma(S) = \gamma\}$.

$$n_g = \sum_{\gamma=0}^{g} N_\gamma(g)$$

## Examples

- $n_0 = 1$ $(\mathbb{N}_0)$ and

$$N_\gamma(0) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 0, & \text{if } \gamma \geq 1. \end{cases}$$

- $n_1 = 1$ $(\mathbb{N}_0 \setminus \{1\})$ and

$$N_\gamma(1) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 0, & \text{if } \gamma \geq 1. \end{cases}$$

- $n_2 = 2$ $(\mathbb{N}_0 \setminus \{1,2\}$ and $\mathbb{N}_0 \setminus \{1,3\})$ and

$$N_\gamma(2) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 1, & \text{if } \gamma = 1 \\ 0, & \text{if } \gamma \geq 2. \end{cases}$$

## F. Torres - 1997

*If $\gamma$ and $g$ are the number of even gaps and the genus of a numerical semigroup $S$, respectively, then $3\gamma \leq 2g$.*

## Remark

If $\gamma$ is even, then

$$\mathbb{N}_0 \setminus (\{2, 4, \ldots, 2\gamma\} \cup \{1, 3, \ldots, \gamma - 1\})$$

is a numerical semigroup with genus $g = \frac{3\gamma}{2}$.

$$n_g = \sum_{\gamma=0}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_\gamma(g)$$

## F. Torres - 1997

*If $\gamma$ and $g$ are the number of even gaps and the genus of a numerical semigroup $S$, respectively, then $3\gamma \leq 2g$.*

## Remark

If $\gamma$ is even, then

$$\mathbb{N}_0 \setminus (\{2, 4, \ldots, 2\gamma\} \cup \{1, 3, \ldots, \gamma - 1\})$$

is a numerical semigroup with genus $g = \frac{3\gamma}{2}$.

$$n_g = \sum_{\gamma=0}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_\gamma(g)$$

**Theorem 1**

*Let $\gamma$ be a nonnegative integer and $g \geq 3\gamma$. Then*

$$N_\gamma(g) = N_\gamma(3\gamma).$$

*Thus, $N_\gamma(g) = N_\gamma(g+1)$, for all $g \geq 3\gamma$.*

**Theorem 2**

*Let $\gamma$ be a nonnegative integer and $g < 3\gamma$. Then*

$$N_\gamma(g) < N_\gamma(3\gamma).$$

### Theorem 1

*Let $\gamma$ be a nonnegative integer and $g \geq 3\gamma$. Then*

$$N_\gamma(g) = N_\gamma(3\gamma).$$

*Thus, $N_\gamma(g) = N_\gamma(g+1)$, for all $g \geq 3\gamma$.*

### Theorem 2

*Let $\gamma$ be a nonnegative integer and $g < 3\gamma$. Then*

$$N_\gamma(g) < N_\gamma(3\gamma).$$

| $g$ / $\gamma$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $n_g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | 1 |
| 1 | 1 | | | | | | | | | | | 1 |
| 2 | 1 | 1 | | | | | | | | | | 2 |
| 3 | 1 | 2 | 1 | | | | | | | | | 4 |
| 4 | 1 | 2 | 4 | | | | | | | | | 7 |
| 5 | 1 | 2 | 6 | 3 | | | | | | | | 12 |
| 6 | 1 | 2 | 7 | 12 | 1 | | | | | | | 23 |
| 7 | 1 | 2 | 7 | 19 | 10 | | | | | | | 39 |
| 8 | 1 | 2 | 7 | 21 | 32 | 4 | | | | | | 67 |
| 9 | 1 | 2 | 7 | 23 | 51 | 33 | 1 | | | | | 118 |
| 10 | 1 | 2 | 7 | 23 | 62 | 91 | 18 | | | | | 204 |
| 11 | 1 | 2 | 7 | 23 | 65 | 142 | 98 | 5 | | | | 343 |
| 12 | 1 | 2 | 7 | 23 | 68 | 174 | 257 | 59 | 1 | | | 592 |
| 13 | 1 | 2 | 7 | 23 | 68 | 192 | 412 | 271 | 25 | | | 1001 |
| 14 | 1 | 2 | 7 | 23 | 68 | 197 | 514 | 678 | 197 | 6 | | 1693 |
| 15 | 1 | 2 | 7 | 23 | 68 | 200 | 570 | 1100 | 793 | 92 | 1 | 2857 |

Notice that

$$n_g = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g).$$

$$n_{g+1} = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g+1) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2(g+1)}{3} \rfloor} N_\gamma(g+1).$$

Theorem 1 states that $N_\gamma(g) = N_\gamma(g+1)$, for $\gamma \leq \frac{g}{3}$.

Notice that

$$n_g = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g).$$

$$n_{g+1} = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g+1) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\left\lfloor \frac{2(g+1)}{3} \right\rfloor} N_\gamma(g+1).$$

Theorem 1 states that $N_\gamma(g) = N_\gamma(g+1)$, for $\gamma \leq \frac{g}{3}$.

Notice that

$$n_g = \sum_{\gamma=0}^{\left\lfloor \frac{g}{3} \right\rfloor} N_\gamma(g) + \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor+1}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_\gamma(g).$$

$$n_{g+1} = \sum_{\gamma=0}^{\left\lfloor \frac{g}{3} \right\rfloor} N_\gamma(g+1) + \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor+1}^{\left\lfloor \frac{2(g+1)}{3} \right\rfloor} N_\gamma(g+1).$$

Theorem 1 states that $N_\gamma(g) = N_\gamma(g+1)$, for $\gamma \leq \frac{g}{3}$.

## Corollary

$$n_g \leq n_{g+1}$$

if, and only if,

$$\sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor+1}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_\gamma(g) \leq \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor+1}^{\left\lfloor \frac{2(g+1)}{3} \right\rfloor} N_\gamma(g+1).$$

| $g$ / $\gamma$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $n_g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | 1 |
| 1 | 1 | | | | | | | | | | | 1 |
| 2 | 1 | 1 | | | | | | | | | | 2 |
| 3 | 1 | 2 | 1 | | | | | | | | | 4 |
| 4 | 1 | 2 | 4 | | | | | | | | | 7 |
| 5 | 1 | 2 | 6 | 3 | | | | | | | | 12 |
| 6 | 1 | 2 | 7 | 12 | 1 | | | | | | | 23 |
| 7 | 1 | 2 | 7 | 19 | 10 | | | | | | | 39 |
| 8 | 1 | 2 | 7 | 21 | 32 | 4 | | | | | | 67 |
| 9 | 1 | 2 | 7 | 23 | 51 | 33 | 1 | | | | | 118 |
| 10 | 1 | 2 | 7 | 23 | 62 | 91 | 18 | | | | | 204 |
| 11 | 1 | 2 | 7 | 23 | 65 | 142 | 98 | 5 | | | | 343 |
| 12 | 1 | 2 | 7 | 23 | 68 | 174 | 257 | 59 | 1 | | | 592 |
| 13 | 1 | 2 | 7 | 23 | 68 | 192 | 412 | 271 | 25 | | | 1001 |
| 14 | 1 | 2 | 7 | 23 | 68 | 197 | 514 | 678 | 197 | 6 | | 1693 |
| 15 | 1 | 2 | 7 | 23 | 68 | 200 | 570 | 1100 | 793 | 92 | 1 | 2857 |

## Conjecture

Let $\gamma$ be a non-negative integer. Then

$$N_\gamma(g) \leq N_\gamma(g+1), \forall g.$$

## Remark

If this Conjecture holds, then $n_g \leq n_{g+1}$ for all $g$.

## Conjecture

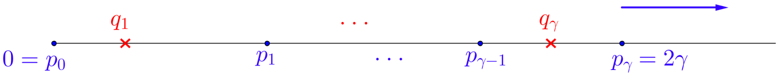
Let $\gamma$ be a non-negative integer. Then

$$N_\gamma(g) \leq N_\gamma(g+1), \forall g.$$

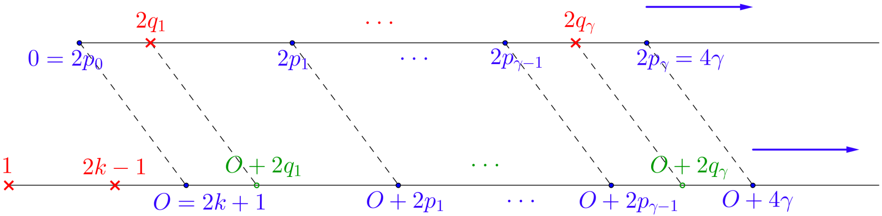
## Remark

If this Conjecture holds, then $n_g \leq n_{g+1}$ for all $g$.

- $\gamma$ - positive integer
- $T = \mathbb{N}_0 \setminus \{q_1, \ldots, q_\gamma\}$ numerical semigroup




- $S = 2 \cdot T \cup (2 \cdot \mathbb{N}_0 + 1) \setminus \{$"suitable choice" of $g - \gamma$ odd numbers$\}$



- "suitable choice" ensures that the final set is closed under addition.
- $S$ is a $\gamma$-hyperelliptic semigroup: even gaps and even non-gaps are determined by $T$.
- $S$ has genus $g$ if, and only if, the number of green points choosen as gaps is $g - \gamma - k$.

**Lemma**

Let $S$ be a $\gamma$-hyperelliptic semigroup of genus $g$ and $O$ the first odd number in $S$. Then

$$2g - 4\gamma + 1 \leq O \leq 2g - 2\gamma + 1.$$

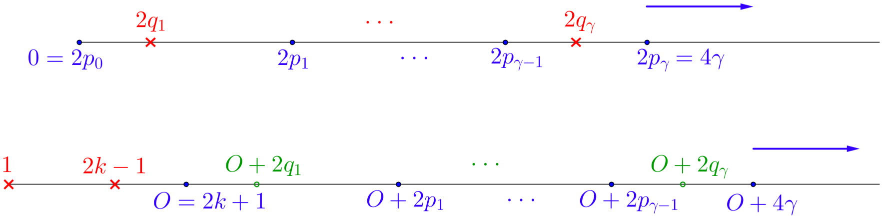
- "suitable choice" ensures that the final set is closed under addition.
- $S$ is a $\gamma$-hyperelliptic semigroup: even gaps and even non-gaps are determined by $T$.
- $S$ has genus $g$ if, and only if, the number of green points choosen as gaps is $g - \gamma - k$.

**Lemma**

Let $S$ be a $\gamma$-hyperelliptic semigroup of genus $g$ and $O$ the first odd number in $S$. Then

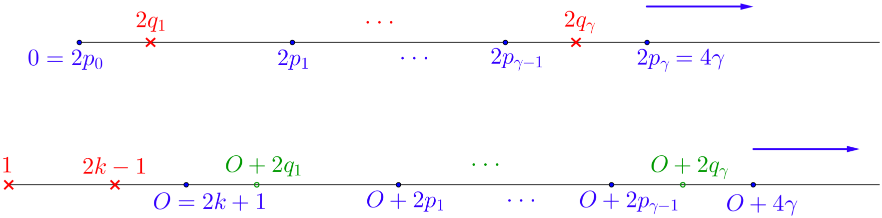
$$2g - 4\gamma + 1 \leq O \leq 2g - 2\gamma + 1.$$

# PROOF (THM 1)




- Numbers $q_i$ must be gaps! (otherwise, $S \ni q_i + q_i = 2q_i \notin S$)
- $g \geq 3\gamma \underbrace{\Rightarrow}_{O \geq 2g - 4\gamma + 1} O \geq 2\gamma + 1$. Hence,
  - $O > q_\gamma \geq q_i$, for all $i$
  - sum of odd elements of $S$ is greater than $4\gamma + 2$.

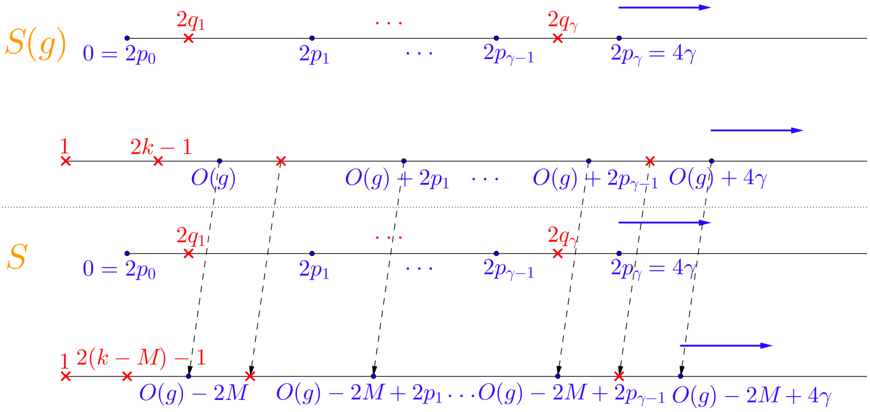
# Proof (Thm 1)




- Numbers $q_i$ must be gaps! (otherwise, $S \ni q_i + q_i = 2q_i \notin S$)
- $g \geq 3\gamma \underbrace{\Rightarrow}_{O \geq 2g-4\gamma+1} O \geq 2\gamma + 1$. Hence,
  - $O > q_\gamma \geq q_i$, for all $i$
  - sum of odd elements of $S$ is greater than $4\gamma + 2$.

# Proof (Thm 1)

- $\mathcal{S}_\gamma(g) := \{S : g(S) = g \text{ and } \gamma(S) = \gamma\}$
- For a fixed $g \geq 3\gamma$, we find a bijection between $\mathcal{S}_\gamma(g)$ and $\mathcal{S}_\gamma(3\gamma)$



# Proof (Thm 1)

- Given $S(g) \in \mathcal{S}_\gamma(g)$, let $O(g)$ be the first odd number of $S(g)$
- Let $M := g - 3\gamma$. Making a translation by $-2M$ only on the odd numbers higher than or equal to $O(g)$, we obtain a NS $S$, such that $O(3\gamma) = O(g) - 2M \geq 2\gamma + 1$ (and this is the first odd number of $S$)
- The even gaps of $S(g)$ and $S$ are the same, as the odd gaps of $S(g)$ and $S$ lower than $O(3\gamma)$. The odd gaps of $S(g)$ and $S$ higher than $O(3\gamma)$ are translated by $-2M$
- Under this construction, we have $g(S) = g - M = 3\gamma$. Hence, $S \in \mathcal{S}_\gamma(3\gamma)$ and $\#\mathcal{S}_\gamma(g) \leq \#\mathcal{S}_\gamma(3\gamma)$
- Similarly (by making a translation by $+2M$), we can verify the other inequality and the result follows.

# A related problem

- $\gamma$ non-negative integer
- For $g \geq 3\gamma$, the sequence $N_\gamma(g)$ is constant and equal to $N_\gamma(3\gamma)$
- A natural task is about the behavior of $f_\gamma := N_\gamma(3\gamma)$

**Lemma**

Let $\gamma$ be a non-negative integer and $M_\gamma := 2^\gamma\left(\frac{\gamma}{2}+1\right)-1$. Then

$$M_\gamma + (n_\gamma - \gamma)\cdot(\gamma+1) \leq f_\gamma \leq M_\gamma + (n_\gamma - \gamma)\cdot 2^\gamma.$$

# A related problem

- $\gamma$ non-negative integer
- For $g \geq 3\gamma$, the sequence $N_\gamma(g)$ is constant and equal to $N_\gamma(3\gamma)$
- A natural task is about the behavior of $f_\gamma := N_\gamma(3\gamma)$

**Lemma**

Let $\gamma$ be a non-negative integer and $M_\gamma := 2^\gamma \left(\frac{\gamma}{2} + 1\right) - 1$. Then

$$M_\gamma + (n_\gamma - \gamma) \cdot (\gamma + 1) \leq f_\gamma \leq M_\gamma + (n_\gamma - \gamma) \cdot 2^\gamma.$$

## Theorem 3

*Let* $\epsilon > 0$. *Then*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{(2\varphi+\epsilon)^\gamma} = 0$$

*and*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

It suggests that the asymptotic behavior of $f_\gamma$ is exponencial of order $\beta^\gamma$, where $2 < \beta \leq 2\varphi$.

## Theorem 3

*Let* $\epsilon > 0$. *Then*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{(2\varphi+\epsilon)^\gamma} = 0$$

*and*

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

It suggests that the asymptotic behavior of $f_\gamma$ is exponencial of order $\beta^\gamma$, where $2 < \beta \leq 2\varphi$.

| $g$ / $\gamma$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $n_g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | 1 |
| 1 | 1 | | | | | | | | | | | 1 |
| 2 | 1 | 1 | | | | | | | | | | 2 |
| 3 | 1 | 2 | 1 | | | | | | | | | 4 |
| 4 | 1 | 2 | 4 | | | | | | | | | 7 |
| 5 | 1 | 2 | 6 | 3 | | | | | | | | 12 |
| 6 | 1 | 2 | 7 | 12 | 1 | | | | | | | 23 |
| 7 | 1 | 2 | 7 | 19 | 10 | | | | | | | 39 |
| 8 | 1 | 2 | 7 | 21 | 32 | 4 | | | | | | 67 |
| 9 | 1 | 2 | 7 | 23 | 51 | 33 | 1 | | | | | 118 |
| 10 | 1 | 2 | 7 | 23 | 62 | 91 | 18 | | | | | 204 |
| 11 | 1 | 2 | 7 | 23 | 65 | 142 | 98 | 5 | | | | 343 |
| 12 | 1 | 2 | 7 | 23 | 68 | 174 | 257 | 59 | 1 | | | 592 |
| 13 | 1 | 2 | 7 | 23 | 68 | 192 | 412 | 271 | 25 | | | 1001 |
| 14 | 1 | 2 | 7 | 23 | 68 | 197 | 514 | 678 | 197 | 6 | | 1693 |
| 15 | 1 | 2 | 7 | 23 | 68 | 200 | 570 | 1100 | 793 | 92 | 1 | 2857 |

| $\gamma$ | $f_\gamma$ | $f_\gamma / f_{\gamma-1}$ | $n_{2\gamma}$ | $f_\gamma / n_{2\gamma}$ |
|---|---|---|---|---|
| 0 | 1 | | 1 | 1 |
| 1 | 2 | 2 | 2 | 1 |
| 2 | 7 | 3,5 | 7 | 1 |
| 3 | 23 | 3,285714 | 23 | 1 |
| 4 | 68 | 2,956522 | 67 | 1,015 |
| 5 | 200 | 2,941176 | 204 | 0,981 |
| 6 | 615 | 3,075 | 592 | 1,039 |
| 7 | 1764 | 2,868292 | 1693 | 1,042 |
| 8 | 5060 | 2,868480 | 4806 | 1,053 |
| 9 | 14626 | 2,890514 | 13467 | 1,086 |
| 10 | 41785 | 2,856899 | 37396 | 1,117 |
| 11 | 117573 | 2,813761 | 103246 | 1,139 |
| 12 | 332475 | 2,827818 | 282828 | 1,176 |
| 13 | 933891 | 2,808905 | 770832 | 1,212 |
| 14 | 2609832 | 2,794579 | 2091030 | 1,248 |

## Conjecture

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{f_{\gamma-1}} = \varphi^2 \approx 2,618$$

and

$$\lim_{\gamma\to\infty} \frac{f_\gamma}{n_{2\gamma}} = C,$$

where $C$ is a constant.

## Remark

Firts item is a consequence of second item.

## Remark

There is a relation between the sequence $f_\gamma$ and the conjecture proposed by M. Bras-Amorós (12). In fact, if $f_\gamma$ is an increasing sequence, then the conjecture is also true.

## Conjecture

$$\lim_{\gamma \to \infty} \frac{f_\gamma}{f_{\gamma-1}} = \varphi^2 \approx 2,618$$

and

$$\lim_{\gamma \to \infty} \frac{f_\gamma}{n_{2\gamma}} = C,$$

where $C$ is a constant.

## Remark

Firts item is a consequence of second item.

## Remark

There is a relation between the sequence $f_\gamma$ and the conjecture proposed by M. Bras-Amorós (12). In fact, if $f_\gamma$ is an increasing sequence, then the conjecture is also true.

- M. Bernardini and F. Torres, Counting numerical semigroups of a given genus via even gaps, *In progress*.
- M. Bras-Amorós, Bounds on the number of numerical semigroups of a given genus, *Journal of Pure and Applied Algebra* **213:6** (2009), 997–1001.
- M. Bras-Amorós, Fibonacci-like behavior of the number of numerical semigroups of a given genus, *Semigroup Forum* **76** (2008), 379–384.
- M. Bras-Amorós, The ordinarization transform of a numerical semigroup and semigroups with a large number of intervals, *Journal of Pure and Applied Algebra* **216** (2012), 2507–2518.
- M. Bras-Amorós and A. de Mier, Representation of numerical semigroups by Dyck Paths, *Semigroup Forum* **75** (2007), 676–681.

S. Elizalde, Improved bounds on the number of numerical semigroups of a given genus, *Journal of Pure and Applied Algebra* **214** (2010), 1862–1873.

N. Kaplan, Counting numerical semigroups by genus and some cases of a question of Wilf, *Journal of Pure and Applied Algebra* **216** (2012), 1016–1032.

F. Torres, On $\gamma$-hyperelliptic numerical semigroups, *Semigroup Forum* **55** (1997), 364-379.

F. Torres, On n-sheeted Covering of Curves and Semigroups which Cannot Be Realized as Weierstrass Semigroups, *Communications in Algebra* **23 (11)** (1995), 4211-4228.

A. Zhai, Fibonacci-like growth of numerical semigroups of a given genus, *Semigroup Forum* **86** (2013), 634–662.

# Thank You!