Almost symmetric property in semigroups and rings

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Some recent papers about almost Gorenstein rings and almost symmetric semigroups:

MR3497969 Prelim Goto, Shiro; Matsuoka, Naoyuki; Taniguchi, Naoki; Yoshida, Ken-ichi; The almost Gorenstein Rees algebras over two-dimensional regular local rings. J. Pure Appl. Algebra 220 (2016), no. 10, 3425-3436.

MR3487216 Prelim Goto, Shiro; Takahashi, Ryo; Taniguchi, Naoki Ulrich ideals and almost Gorenstein rings. Proc. Amer. Math. Soc. 144 (2016), no. 7, 2811-2823.

MR3484144 Prelim Moscariello, Alessio; On the type of an almost Gorenstein monomial curve. J. Algebra 456 (2016), 266-277.

MR3484141 Prelim Higashitani, Akihiro; Almost Gorenstein homogeneous rings and their h-vectors. J. Algebra 456 (2016), 190-206.

MR3478852 Pending Matsuoka, Naoyuki; Murai, Satoshi Uniformly Cohen-Macaulay simplicial complexes and almost Gorenstein? simplicial complexes. J. Algebra 455 (2016), 14-31. 13F55

MR3461066 Pending Goto, Shiro; Matsuoka, Naoyuki; Taniguchi, Naoki; Yoshida, Ken-ichi The almost Gorenstein Rees algebras of parameters. J. Algebra 452 (2016), 263-278. 13H10 (13A30 13H15)

MR3313502 Reviewed Goto, Shiro; Takahashi, Ryo; Taniguchi, Naoki Almost Gorenstein rings-towards a theory of higher dimension. J. Pure Appl. Algebra 219 (2015), no. 7, 2666-2712. (Reviewer: Aryampilly V. Jayanthan) 13H10 (13A30 13H15)

MR3019262 Reviewed Goto, Shiro; Matsuoka, Naoyuki; Phuong, Tran Thi Almost Gorenstein rings. J. Algebra 379 (2013), 355-381. (Reviewer: Siamak Yassemi) 13H10

MR3401797 Reviewed Strazzanti, F. One half of almost symmetric numerical semigroups. Semigroup Forum 91 (2015), no. 2, 463-475.

MR3169635 Reviewed Rosales, J. C.; Garcia-Sanchez, P. A. Constructing almost symmetric numerical semigroups from irreducible numerical semigroups. Comm. Algebra 42 (2014), no. 3, 1362-1367.

MR3016268 Reviewed Nari, Hirokatsu Symmetries on almost symmetric numerical semigroups. Semigroup Forum 86 (2013), no. 1, 140-154.

MR2837144 Reviewed Fel, Leonid G. Duality relation for the Hilbert series of almost symmetric numerical semigroups. Israel J. Math. 185 (2011), 413-444.

Let's start with the num. semigroup context. S is a *numerical semigroup*.

$M = S \setminus \{0\}$ is the *maximal ideal* of S

f_S is the *Frobenius number* of S, that is the greatest integer which does not belong to S.

A *relative ideal* of S is a nonempty subset I of $\mathbb{Z}$ (which is the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$.

If I, J are relative ideals of S, then the following is a relative ideals too:

$$I - J = \{z \in \mathbb{Z} \mid z + J \subseteq I\}$$

I is *bidual* whenever $I = S - (S - I)$

Define

$$I \sim J$$

if $J = z + I$, for some $z \in \mathbb{Z}$.

- If $I \sim J$ then I is bidual if and only if J is bidual.
- In each equivalence class of ideals of S we can choose an ideal I, $S \subset I \subset \mathbb{N}$

The *blowup* of an ideal I of S is

$$B(I) = \bigcup_{n \in \mathbb{N}} (nI - nI) = (nI - nI), \text{ for } n >> 0$$

If $I \sim J$, then $B(I) = B(J)$

If $S \subset I \subset \mathbb{N}$, then

$(nI - nI) = ((n+1)I - (n+1)I) \Leftrightarrow nI = (n+1)I \Leftrightarrow nI$ is a semigroup

Thus, if $S \subset I \subset \mathbb{N}$, then

$$I \subset 2I \subset \cdots \subset nI = B(I)$$

and $B(I)$ is the smallest semigroup containing I.

The multiplicity e is the smallest positive integer in S

the Apery set is $\mathrm{Ap}_e(S) = S \setminus (e + S)$

The minimal set of generators of an ideal I is $I \setminus (M + I)$ and the cardinality of the minimal set of generators of M i.e. the cardinality of $M \setminus 2M$ is the embedding dimension of S.

$M \setminus 2M$ is also the set of minimal generators of the semigroup S.

A particular relative ideal of S plays a special role. It is the *canonical ideal*

$$\Omega = \{f_S - x \mid x \in \mathbb{Z} \setminus S\}$$

It is

$$S \subseteq \Omega \subseteq \mathbb{N}$$

and the blowup $B(\Omega)$ is the smallest semigroup containing Ω

Facts:

- For each ideal I of S:

$$(\Omega - I) = \{f_S - x \mid x \in \mathbb{Z} \setminus I\}$$

- $\Omega - (\Omega - I) = I$, for each ideal I of S.

  The type t of S is the cardinality of the minimal set of generators of Ω, Card $(\Omega \setminus (\Omega + M))$ and

- $t = \mathrm{Card}((S - M) \setminus S)$ (The elements of $(S - M) \setminus S$ are called the *pseudo-Frobenius numbers* of S)

- $1 \leq t \leq e - 1$

S is called *symmetric* if one of the equivalent conditions holds:

1. $\Omega = S$
2. $B(\Omega) = S$
3. $f_S - x \in S$ for each $x \in \mathbb{Z} \setminus S$
4. Each ideal I of S is bidual, i.e. $S - (S - I) = I$

S is called *almost symmetric* if one of the equivalent conditions holds:

1. $\Omega \subseteq M - M$
2. $B(\Omega) \subseteq M - M$
3. $f_S - x \in B(\Omega)$ for each $x \in \mathbb{Z} \setminus S$
4. Each ideal I of $B(\Omega)$ is S-bidual, i.e. $S - (S - I) = I$

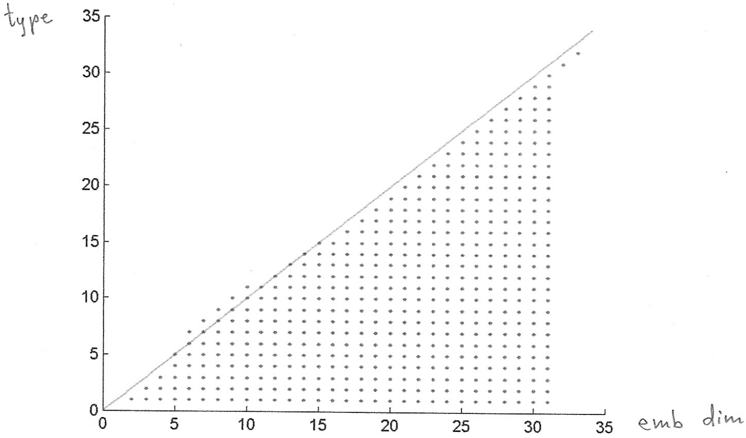
If S is almost symmetric and non symmetric, then
$B(\Omega) = \Omega \cup \{f_S\} = M - M$

A construction to get all the almost symmetric numerical semigroups with prescribed Frobenius number:

Theorem (Rosales - García-Sánchez) T is almost symmetric if and only if there exists an irreducible numerical semigroup S with $f_S = f_T$ such that $T = S \setminus A$, where $A \subseteq [f_S/2, f_S]$ is a set of minimal generators of S such that $x + y - f_S \notin T$ for any $x, y \in A$. In this case $t(T) = 2|A| + t(S)$.

They do not compare the embedding dimensions of S and T.

I think that that construction, implemented with numericalsgps package permitted to A. Moscariello to drow this picture



and to conjecture that in almost symmetric semigroups the type may not be too hight. He investigates possible bounds for the type of an almost symmetric semigroup in terms of the embedding dimension and proves that

Theorem (Moscariello) The type of an almost symmetric semigroup of embedding dimension 4 is at most 3.

Recall that in general (no almost symmetric hypothesis)
emb dim 2 $\Rightarrow t = 1$
emb dim 3 $\Rightarrow t \leq 2$
emb dim$> 3 \ \Rightarrow$ no upper bound for the type

How to find such a possible bound?

Can Nari simmetries help?

Theorem (Nari) Let $\{f_1 < f_2 < \cdots < f_t = f_S\}$ be the pseudo-Frobenius numbers of S. Then S is almost symmetric if and only if

$$f_i + f_{t-i} = f_S$$

for all $i, 1 \leq i \leq t-1$

Let (S, M) be a semigroup and let $a \in \mathbb{N}$, $a > 0$ such that $T = \{0\} \cup \{M + a\}$ is also a semigroup (notation: $T = S + a$). Then $f_T = f_S + a$ and $t(T) = t(S) + a$.

Lemma Let $T = S + a$ be semigroups.
Then:

1. $\Omega_S \setminus \{f_T\} \subset \Omega_T$
2. $\Omega_T \setminus \{f_S\} \subset \Omega_S$

Proposition Let $T = S + a$ be semigroups. Then S is almost symmetric if and only if T is almost symmetric.

Proof.
S is almost symmetric if and only if $\Omega_S \subset M - M$ and T is almost symmetric if and only if $\Omega_T \subset (M + a) - (M + a) = M - M$. Since always $f_S, f_T \in M - M$, we finish by point (2) of the Lemma above. □

Example. $S = \langle 11, 14, 18, 20, 21, 23, 24, 27, 30\rangle$ is almost symmetric (of type 8 and embedding dimension 9). If $a = 5$:

$T = S + a = \langle 16, 19, 23, 25, 26, 27, 28, 29, 30, 33, 34, 36, 37, 40\rangle$ which is also almost symmetric (of type $8 + 5 = 13$ and embedding dimension $9 + 5 = 14$).

Question. Given a semigroup S, can we find any regularity in the class of semigroups $S + a$, $a \in \mathbb{Z}$? Of course there is one of smallest multiplicity.

Is always embdim$(S + a)$ =embdim$(S) + a$, as in the example above? (Yes, if S is of max emb dim)

F. Strazzanti's remark (two days ago):

Let S be a numerical semigroup, $I \subset S$ an ideal and $b \in S$ an odd integer. Define the *duplication* of S with respect to I as

$$S \bowtie^b I := (2 \cdot S) \cup (2 \cdot I + b)$$

where $2 \cdot S = \{2s;\ s \in S\}$

There is a formula for the type of $S \bowtie^b I$ in terms of the type of S (D'Anna - Strazzanti). In particular, if S is almost symmetric of type t and embdim ν, then $S \bowtie^b M$ is almost symmetric of

type=$2t+1$ and embdim=2ν

So, starting for example with $S = \langle 14, 15, 17, 19, 20 \rangle$

which is almost symmetric with $t = 5$ and $\nu = 5$, we get

$$T = S \bowtie^{15} M$$

which is almost symmetric with

$t(T) = 2t + 1 = 11$ and $\nu(T) = 2\nu = 10$, Iterating n times such duplications, we get an almost symmetric semigroup T_n with

type $t(T_n) = 2^n(t+1) - 1$ and embdim $\nu(T_n) = 2^n \nu$

So for $n >> 0$ also the difference type $-$ embdim $>> 0$ and a bound of the form

$$\text{type} \leq \text{embdim} + \text{constant}$$

does not exist.

However

$$t(T_n) = 2^n(t+1) - 1 < 2^n(t+1) = 2^n \nu \frac{(t+1)}{\nu} = \nu(T_n)\frac{(t+1)}{\nu}$$

Question: type $<$ 2 embdim in almost symmetric semigroups??

Let (R, m) be a local Cohen Macaulay ring possessing a canonical module ω.

R has a canonical module if and only if $R \cong G/I$ for some Gorenstein ring G. In particular any quotient of a regular ring has a canonical module.

If $\dim R = 1$, then a canonical module is a fractional regular ideal ω of R such that $\omega : (\omega : I) = I$, for all fractional regular ideals I of R.

If $\dim R > 1$, then $\mathrm{Hom}_R(-, \omega)$ is a dualizing functor on the category of maximal Cohen Macaulay modules and we have

$$\mathrm{Hom}_R(\mathrm{Hom}_R(M, \omega), \omega) \cong M$$

for each maximal CM module M.

If the canonical module exists and is isomorphic to R, then R is a *Gorenstein ring*. So in dimension one, R is Gorenstein if and only if each ideal I is divisorial:

$$\omega : (\omega : I) = R : (R : I) = I$$

If (R, m) is a local one-dimensional CM ring with finite integral closure (i.e. *analytically unramified*), it has a canonical ideal ω which can be supposed

$$R \subseteq \omega \subseteq \overline{R}$$

In Journal of Algebra (1997), besides *almost symmetric numerical semigroups* , we (V.B. - R. Fröberg) defined a one-dimensional analytically unramified ring R to be *almost Gorenstein* if one of the equivalent conditions are satisfied

1. $\omega \subseteq m : m$
   (we have $\Omega \subseteq M - M$ for almost symm. semigroups)
2. $m = \omega m$
3. $\omega : m = m : m$ (supposing R not a DVR)
4. $m\omega \subset R$

R is almost Gorenstein if and only if $m\omega \subset R$.
Thus for a one-dimensional analytically unramified ring we have an exact sequence of R-modules

$$0 \to R \to \omega \to \omega/R \to 0$$

and R is almost Gorenstein if and only $m\omega \subset R$, i.e. if and only if $m\,(\omega/R) = 0$.
Thus R is almost Gorenstein if and only if there is an exact sequence of R-modules

$$0 \to R \to \omega \to C \to 0$$

such that $mC = 0$
In particular, if R is Gorenstein, $R = \omega$ and $C = 0$.

S. Goto - N. Matsuoka - T. Thi Phuong. Almost Gorenstein rings. (J. Algebra 2013) extended this definition to any one-dimensional CM ring which has a canonical ideal. If the residue field is finite not always it's possible to have ω between R and its integral closure, so they prefer to talk of the canonical ideal as an m-primary ideal.

More recently S. Goto, R. Takahashi and N. Taniguchi (J. Pure Appl. Alg. 2015) extended this definition to local CM rings of any dimension which have a canonical module.

The smart generalized definition of almost Gorenstein rings given in that paper has a form similar to our definition in case of a one-dimensional analytically unramified ring.

In fact a Cohen-Macaulay local ring (R, m) of any Krull dimension d, possessing a canonical module ω is defined almost Gorenstein if there is an exact sequence of R-modules

$$0 \to R \to \omega \to C \to 0$$

such that $\mu_R(C) = e^0_m(C)$, where $\mu_R(C) = \ell_R(C/mC)$ is the number of generators of C and $e^0_m(C)$ is the multiplicity of C with respect to m.

Among several results they prove that the notion is stable by idealization. More precisely:
If E is an R-module, the Nagata idealization of E over R, $R \ltimes E$, is $R \oplus E$, with the product defined:

$$(r, i)(s, j) = (rs, rj + si)$$

Theorem (S. Goto, N. Matsuoka, T. Phuong, + S. Goto, R. Takahashi, N. Taniguchi)
Let (R, m) be a CM local ring. Then the following conditions are equivalent:
(1) $R \ltimes I$ is an almost Gorenstein ring
(2) Some conditions on I are verified (“Goto conditions")
In particular:
(1) $R \ltimes m$ is an almost Gorenstein ring
(2) R is an almost Gorenstein ring

It’s a way to construct many examples of analytically ramified almost Gorenstein rings, that are not Gorenstein.

Marco and Marco (D'Anna and Fontana) introduced the *amalgamated duplication* of a ring R along an R-module E, $R \bowtie E$ as $R \oplus E$, with the product defined:

$$(r, i)(s, j) = (rs, rj + si + ij)$$

$R \bowtie E$ can be reduced and it is always reduced if R is a domain. On the contrary the Nagata idealization $R \ltimes E$ is an extension of R containing an ideal $\overline{E}$ such that $\overline{E}^2 = 0$. So $R \ltimes E$ is not a reduced ring.

Let R be a commutative ring with unity and let I be a proper ideal of R. In two joint papers with M. D'Anna and F. Strazzanti we studied the family of quotient rings

$$R(I)_{a,b} = \mathcal{R}_+/(I^2(t^2 + at + b)),$$

where $\mathcal{R}_+$ is the Rees algebra associated to the ring R with respect to I (i.e. $\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n$) and $(I^2(t^2 + at + b))$ is the contraction to $\mathcal{R}_+$ of the ideal generated by $t^2 + at + b$ in $R[t]$.

Thus

$$R(I)_{a,b} = \{\overline{r+it}; r \in R,\ i \in I\}$$

and the product

$$(\overline{r+it})(\overline{s+jt}) = \overline{rs + (rj+si)t + ijt^2}$$

is defined mod the ideal $I^2(t^2+at+b)$. So, if $t^2+at+b = t^2$, we get the idealization: $\mathcal{R}_+/(I^2t^2) \cong R \ltimes I$. In fact

$$(\overline{r+it})(\overline{s+jt}) = \overline{rs + (rj+si)t}$$

If $t^2+at+b = t^2 - t$, we get the duplication: $\mathcal{R}_+/(I^2(t^2-t)) \cong R \bowtie I$. In fact:

$$(\overline{r+it})(\overline{s+jt}) = \overline{rs + (rj+si+ij)t}$$

We proved:

- The ring extensions $R \subseteq R(I)_{a,b} \subseteq R[t]/(f(t))$ are both integral and the three rings have the same Krull dimension and the last two have the same total ring of fractions and the same integral closure;
- R is a Noetherian ring if and only if $R(I)_{a,b}$ is a Noetherian ring for all $a, b \in R$ if and only if $R(I)_{a,b}$ is a Noetherian ring for some $a, b \in R$;
- $R = (R, \mathfrak{m})$ is local if and only if $R(I)_{a,b}$ is local. In this case the maximal ideal of $R(I)_{a,b}$ is $\mathfrak{m} \oplus I$ (as R-module), i.e. $\{m + it; m \in \mathfrak{m}, i \in I\}$

In the local case:

- $R(I)_{a,b}$ is CM if and only if R is a CM ring and I is a maximal CM R-module. In particular, the Cohen-Macaulayness of $R(I)_{a,b}$ depends only on the ideal I.
- $R(I)_{a,b}$ is Gorenstein if and only if R is Cohen-Macaulay and I is a canonical ideal of R.
- The almost Gorensteinness of $R(I)_{a,b}$ does not depend on a and b.

Since the idealization $R \ltimes I = R(I)_{0,0}$ is a particular case of our construction we can apply Goto et al. result:

Theorem (B. - D'Anna - Strazzanti) $R(I)_{a,b}$ is almost Gorenstein if and only if “Goto conditions" are satisfied

Thank you for the attention