Canonical bases for one dimensional algebras and modules

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IMNS,Levico Terme, 2016

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Main objectives

1. Study the singularities of plane curves parametrizeb by polynomials over $x = x(t), y = y(t)$ over $\mathbb{C}$.

2. Study the embedding of such curves in $\mathbb{C}^2$.

$\mathbb{K}$ is a field

$\mathbf{A} = \mathbb{K}[x_1(t), \ldots, x_n(t)] \subset \mathbb{K}[t]$ with $l(\mathbb{K}[t]/\mathbf{A}) < +\infty$.

$F(X_1, \ldots, X_n) \longmapsto d(F) =$ the degree in t of $F(x_1(t), \ldots, x_n(t))$

The set $\Gamma(\mathbf{A}) = \{d(F), F \in \mathbb{K}[X_1, \ldots, X_n]\}$ is a numerical semigroup.

We say that $b_1(t), \ldots, b_r(t) \in \mathbf{A}$ is a canonical basis of $\mathbf{A}$ if $\Gamma(\mathbf{A}$ is generated by $d(b_1), \ldots, d(b_r)$. We have the following:

Theorem (.., P.A. Garcia, V. Micale) Starting from $x_1(t), \ldots, x_n(t)$, we can compute a canonical basis of $\mathbf{A}$ in an algorithmic way.

Main idea: Division Theorem

$f(t) \in \mathbb{K}[t]$, suppose that $f, x_1, \ldots, x_n$ are monic. Write $x_i(t) = t^{d_i} + \ldots$ and $f = c_d t^d + \ldots$.

1. If $d \in \langle d_1, \ldots, d_n \rangle$ then $d = \sum_{i=1}^{n} a_i d_i$. We set

$$f^1 = f - x_1(t)^{a_1} \cdots x_n(t)^{a_n}$$

in such a way that $d(f^1) < d$.

2. Otherwise, we set $f^1 = f - t^d$ and $r^1 = t^d$.

In both cases we restart with f^1.

In conclusion, $f = g + r$ and

$g \in \mathbf{A}$, $d(g) \leq d$, $d(r) \notin \langle d_1, \ldots, d_n \rangle$ and $d(r) \leq d$.

Algorithm

$\mathbf{A} = \mathbb{K}[x_1(t), \ldots, x_n(t)]$ with $x_i(t) = t^{d_i} + \ldots$

1. Compute a system of generators of the Kernel of

$$\phi : \mathbb{K}[X_1, \ldots, X_n] \longmapsto \mathbb{K}[t], \phi(X_i) = t^{d_i}$$

Let $K_1, \ldots, K_s$ be such a system.

2. Compute $P_i = K_i(x_1(t), \ldots, x_n(t)$.

3. Divide $P_1, \ldots, P_s$ by $x_i(t), \ldots, x_n(t)$.

4. If all remainders are 0 then $x_1, \ldots, x_n$ is a canonical basis of $\mathbf{A}$, otherwise we add all nonzero remainders and we restart with the new system.

Example $x_1(t) = t^4, x_2(t) = t^6 + t$, $\mathbf{A} = \mathbb{K}[x_1(t), x_2(t)]$:

1. $\phi : \mathbb{K}[X_1, X_2] \longmapsto \mathbb{K}[t], \phi(X_1) = t^4, \phi(X_2) = t^6$

a) $\mathrm{Ker}(\phi) = (X_2^2 - X_1^3)$

b) $x_2(t)^2 - x_1(t)^3 = 2t^7 + t^2$

c) $7 \notin < 4, 6 >$

we add $x_3(t) = t^7 + \frac{1}{2}t^2$: $\mathbf{A} = \mathbb{K}[x_1(t), x_2(t), x_3(t)]$.

2. $\phi_1 : \mathbb{K}[X_1, X_2, X_3] \longmapsto \mathbb{K}[t], \phi_1(X_1) = t^4, \phi(X_2) = t^6$, $\phi(X_3) = t^7$

a) $\mathrm{Ker}(\phi_1) = (X_2^2 - X_1^3, X_3^2 - X_1^2 X_2)$

b) $x_3(t)^2 - x_1(t)^2 x_2(t) = \frac{1}{4}t^4 = \frac{1}{4}x_1(t)$.

3. In conclusion, $\{x_1(t), x_2(t), x_3(t)\}$ is a canonical basis of $\mathbf{A}$ (hence $\Gamma(\mathbf{A}) = \langle 4, 6, 7 \rangle$).

Modules and ideals

$\mathbf{A} = \mathbb{K}[x_1(t), \ldots, x_n(t)]$, x_i is monic $\forall i$, $\Gamma(\mathbf{A}) = \langle d(x_1), \ldots, d(x_n) \rangle$.

$F_1(t), \ldots, F_s(t) \in \mathbb{K}[t]$

$\mathbf{M} = F_1\mathbf{A} + \ldots + F_s\mathbf{A}$

$\mathbf{I} = \cup_{F \in \mathbf{M}} d(F) + \Gamma(\mathbf{A})$ is an ideal of $\Gamma(\mathbf{A})$

Problem How to compute a system of generators of **I**? In order to answer the problem:

I. We need to generalize the notion of relations.

II. We need a division theorem.

Relations

$F_i = t^{a_i} + \ldots$ for all i, $S = \Gamma(\mathbf{A})$

$\psi : \mathbf{A}^s \longmapsto \mathbf{M} = t^{a_1}\mathbf{A} + \ldots + t^{a_s}\mathbf{A}$

$\psi(g_1, \ldots, g_s) = g_1 t^{a_1} + \ldots + g_s t^{a_s}$. What is $\mathrm{Ker}(\psi)$?

Given $a, b \in \mathbb{N}$, we set $a \leq_S b$ if $b - a \in S$.

$c \in \min_{\leq_S}(a + S) \cap (b + S)$
$a + s_1 = b + s_2 = c, t^{s_1}t^a - t^{s_2}t^b = 0$
$R(a, b) = \{(s_1, s_2), ...\}$

$s = 2 : \{(t^{s_1}, -t^{s_2}), (s_1, s_2) \in R(a_1, a_2)\}$ generates $\mathrm{Ker}(\psi)$. More generally:

Theorem $\{(0, \ldots, t^{s_i}, \ldots, -t^{s_j}, 0, \ldots, 0), (a_i, a_j) \in R(a_i, a_j)\}$ generates $\mathrm{Ker}(\psi)$.

Example

$\mathbf{A} = \mathbb{K}[t^4, t^6 + t] = \mathbb{K}[t^4, t^6 + t, t^7 + \frac{1}{2}t^2]$

$S = \Gamma(\mathbf{A}) = \langle 4, 6, 7 \rangle$

$\mathbf{M} = F_1\mathbf{A} + F_2\mathbf{A}$ with $F_1 = t^3$ and $F_4 = t^4$.

$3 + S = \{3, 7, 9, 10, 11, 13, \longmapsto\}$ and
$4 + S = \{4, 8, 10, 11, 12, 14, \longmapsto\}$

$(3 + S) \cap (4 + S) = \{10, 11, 14, \longmapsto\}$, $\min_{(3+S)\cap(4+S)} = \{10, 11\}$

$R(3, 4) = \{(7, 6), (8, 7)\}$ with $3 + 7 = 4 + 6 = 10$ and
$3 + 8 = 4 + 7 = 11$. Hence

$\mathrm{Ker}(\psi) = ((t^7, -t^6), (t^8, -t^7))$

Division Theorem

Same notations: $F_1, \ldots, F_r \in \mathbb{K}[t]$, $\mathbf{M} = \sum_{i=1}^{r} F_i \mathbf{A}$.

Theorem Given $F \in \mathbb{K}[t], F \neq 0$, there exist $g_1, \ldots, g_r, r \in \mathbf{A}$ such that:

1. $F = \sum_{i=1}^{r} g_i F_i + R$.
2. For all $i \in \{1, \ldots, r\}$, if $g_i \neq 0$ then $d(g_i) + d(F_i) \leq d(F)$.
3. If $R \neq 0$ then $d(R) \leq d(F)$ and $d(R) \in \mathbb{N} - \cup_{i=1}^{r}(d(F_i) + d(\mathbf{A}))$.

Algorithm

$\mathbf{A} = \mathbb{K}[x_1(t), \ldots, x_n(t)]$, with $x_i(t) = t^{d_i} + \ldots$ and

$S = \Gamma(\mathbf{A}) = \langle d_1, \ldots, d_n \rangle$

$\mathbf{M} = F_1\mathbf{A} + \ldots + F_s\mathbf{A}$, $F_i(t) = t^{a_i} + \ldots$.

$E = S(F_1, \ldots, F_s) = \cup_{i,j} R(a_i, a_j)$

1. Choose $(s_i, s_j) \in R(a_i, a_j) \subseteq E : s = s_i + a_i = s_j + a_j$
2. Choose g_i, g_j monic in $\mathbf{A}$ such that $g_i = t^{s_i} + ..., g_j = t^{s_j} + ...$
3. Divide $g_i F_i - g_j F_j$ (whose degree is $< s$) by $F_1, \ldots, F_s$.
4. Do this for all elements in E. If all the remainders are 0 then $\mathbf{I} = \cup_{i-1}^{s}(a_i + S)$. Otherwise we add the non zero remainders and we restart with the new system....

Example

$\mathbf{A} = \mathbb{K}[t^6+t, t^4] : \{f_1 = t^6+t, f_2 = t^4, f_3 = t^7 + \frac{1}{2}t^2\}$ is a canonical basis of $\mathbf{A}$.

$\mathbf{M} = F_1\mathbf{A} + F_2\mathbf{A}$ with $F_1 = t^3$ and $F_4 = t^4$, $I = d(\mathbf{M})$

$R(3,4) = \{(7,6),(8,7)\} : 3+7 = 4+6 = 10 , 3+8 = 4+7 = 11.$

$f_3F_1 - f_1F_2 = \frac{1}{2}t^5$ and $5 \notin (3+S) \cup (4+S)$

$f_2^2F_1 - f_3F_2 = \frac{1}{2}t^6$ and $6 \notin (3+S) \cup (4+S)$

We set $F_3 = t^5, F_4 = t^6$ and we restart with $\{F_1, F_2, F_3, F_4\}$.

$\min_{\leq S}(3,5) = \{9,11\}$, $R(3,5) = \{(6,4),(7,6)\}$...

We can verify that $\{3,4,5,6\}$ is a system of generators of I.

Ideal of differentials

$\mathbf{A} = \mathbb{K}[x_1(t), \ldots, x_n(t)], S = \Gamma(\mathbf{A})$

$\mathbf{M} = x_1'(t)\mathbf{A} + \ldots + x_n'(t)\mathbf{A}, I = \cup_{F \in \mathbf{M}}(d(F) + S)$

If $s \in S$ then $s - 1 \in I$. The set $\{s - 1, s \in S\}$ is called the set of exact elements. The other elements in I (if any) are called non exact elements (NE(I) for short). Their cardinality is denoted ne(I). Let $F(S)$ be the Frobinus number of S. We have

1. For all $s \geq F(S)$, s is exact.
2. If s is non exact then $s + 1$ is a gap of S.
3. $ne(I) \leq g(S)$ where $g(S)$ denotes the genus of S

Polynomial plane curves

$\mathbf{A} = \mathbb{K}[x(t), y(t)]$ and $\mathbb{K}$ is an algebraically closed field of characteristic 0.

$x(t) = t^n + a_1 t^{n-1} + \ldots, y(t) = t^m + b_1 t^{m-1} + \ldots$ with $n > m$ and $m \nmid n$.

$f(X, Y) = Y^n + c_1(X) Y^{n-1} + \ldots + c_n(X)$ is the minimal polynomial of $(x(t), y(t))$. We say that $V(f) \in \mathbb{K}^2$ is a polynomial curve.

Theorem 1. f has one place at infinity and $S = \Gamma(\mathbf{A})$ is a free numerical semigroup.

2. $F(S) + 1 = \mu(f) = \mathrm{rank}_{\mathbb{K}} \dfrac{\mathbb{K}[X, Y]}{(f_x, f_y)}$ (The Milnor number of f).

3. $g(S)$ is the geometric genus of a smooth curve $f + \lambda, \lambda \in \mathbb{K}$.

Polynomial plane curves

Let $\nu(f) = \mathrm{rank}_{\mathbb{K}} \dfrac{\mathbb{K}[X, Y]}{(f, f_x, f_y)}$: $\mu(f)$ is called the Turina number of f.

Proposition 1. $\nu(f) \leq \mu(f)$

2. $\nu(f) \geq \dfrac{\mu(f)}{2}$ (hence $g(S) \leq \nu(f) \leq F(S) + 1 = 2g(S)$).

3. $\mu(f) = \nu(f) + ne(I)$

Theorem $\mu(f) = \nu(f)$ if and only if $ne(I) = 0$ if and only if $\gcd m, n) = 1$ and $f \sim Z^n - W^m$ if and only if $V(f + \lambda)$ is a smooth curve for all $\lambda \neq 0$ (and $V(f)$ has only one singularity which is a cusp)

Polynomial plane curves

Theorem $ne(I) = \nu(f) = g(S)$ if and only if $V(f)$ has $g(S)$ singularities and all of them are nodes if and only if $I = \mathbb{N}$.

Remark $0 < ne(I) < g(S)$ if and only if $g(S) < \nu(f) < 2g(S) = \mu(f)$

Proposition If $ne(I) = 1$ then $\gcd(m, n) = 1$ and $(m, n) = (2, 2k + 1), (3, 4), (3, 5)$

Proposition If $ne(I) = 2$ then the embedding dimension of S is eith two ($\gcd(m, n) = 1$) or three and ....

Cases when $ne(I) = g(S) - 1$

Remark 1. $0 < ne(I) < g(S)$ implies that $V(f)$ has cusps and nodes
2. $ne(I) = 0$ implies that $V(f)$ has only cusps (one)
3. $ne(I) = g(S)$ implies that $V(f)$ has only nodes.

Example

$\mathbf{A} = \mathbb{K}[x(t) = t^3, y(t) = t^4 - 2t^2]$: $S = \Gamma(\mathbf{A}) = \langle 3, 4 \rangle$.

$\mathbf{M} = t^2\mathbf{A} + (t^3 - t)\mathbf{A}$. We set $F_1(t) = t^2, F_2(t) = t^3 - t$.

$\min_{\leq_S}(2 + S) \cap (3 + S) = \{6, 11\}$

$R(2, 3) = \{(4, 3), (9, 8)\}$

$x_1 F_2 - x_2 F_1 = -t^4$ and $(x_1^3 F_1 - x_2^2 F_2 = 5t^9 - 8t^7 + 4t^5$. The division of these elements gives t^4 and t as remainders (up to constants).

We set $F_3(t) = t^4, F_4(t) = t$ and we restart with $\{F_1, F_2, F_3, F_4\}$. We verify that this system generates $I = d(\mathbf{M})$. Hence $I = \mathbb{N}^*$.

$\mathrm{Ne}(I) = \{1, 4\}$ hence $\mathrm{ne}(I) = 2$ and $\mu(f) = 6 = 4 + \mathrm{ne}(I)$, hence $\nu(f) = 4$.

All the algorithms are implemented in GAP.

THANK YOU