Apéry Sets and Feng-Rao Numbers over Telescopic Numerical Semigroups

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Join work with

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Feng-Rao Number

- Γ a numerical semigroup, the generalized Feng-Rao distance is:

$$\delta_{FR}^{r}(s) = \min\{\# D_\Gamma(s_1,\dots,s_r) \mid s \le s_1 < \dots < s_r,\ s_i \in \Gamma\}$$

$$D_\Gamma(s_1,\dots,s_r) = \bigcup_i D(s_i)$$

$$D_\Gamma(s) = \{n \in \Gamma \mid s-n \in \Gamma\}$$

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- The Feng-Rao number $E(\Gamma, r)$ is the constant such that

$$\delta_{FR}^{r}(s) = s + 1 - 2g + E(\Gamma, r)$$

for $s \geq 2c - 1$

Feng-Rao Number and Apéry sets

- For $r=2$ we have

$$E(\Gamma,2)=\min\{\#Ap(\Gamma,x) \mid 1\leq x\leq m(\Gamma)\}$$

where

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- Goal: Compute $E(\Gamma,2)$ for certain numerical semigroups.

More about Apéry sets

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$$i + nx \quad n \in \mathbb{Z}$$

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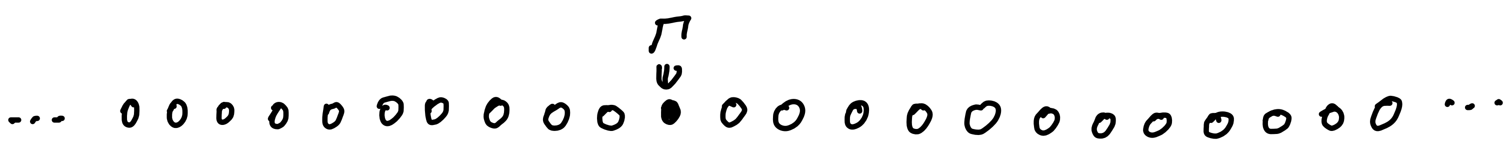

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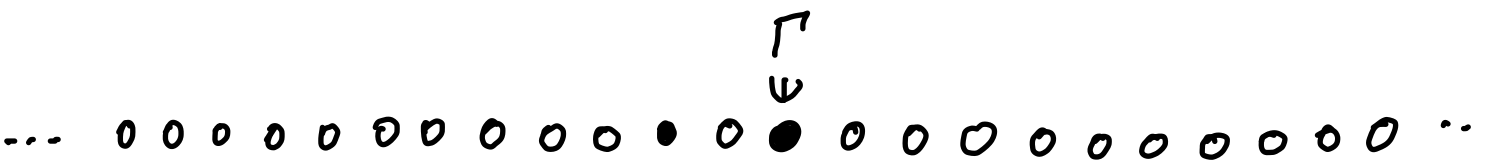

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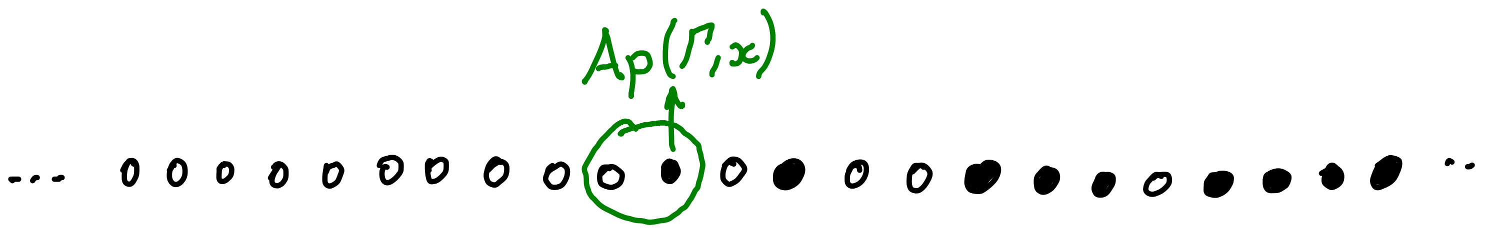

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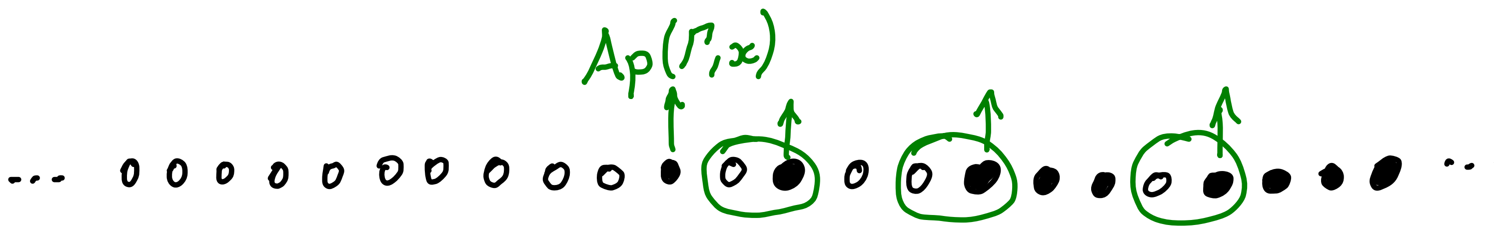

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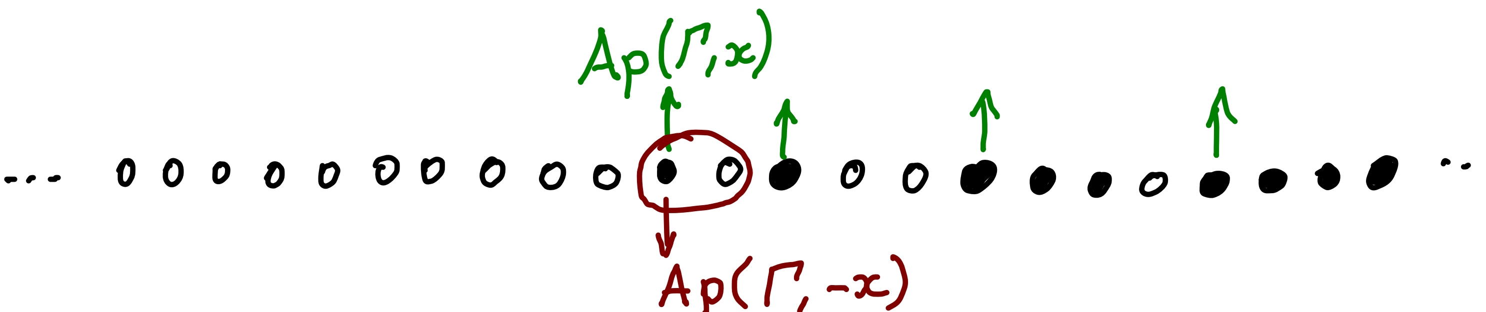

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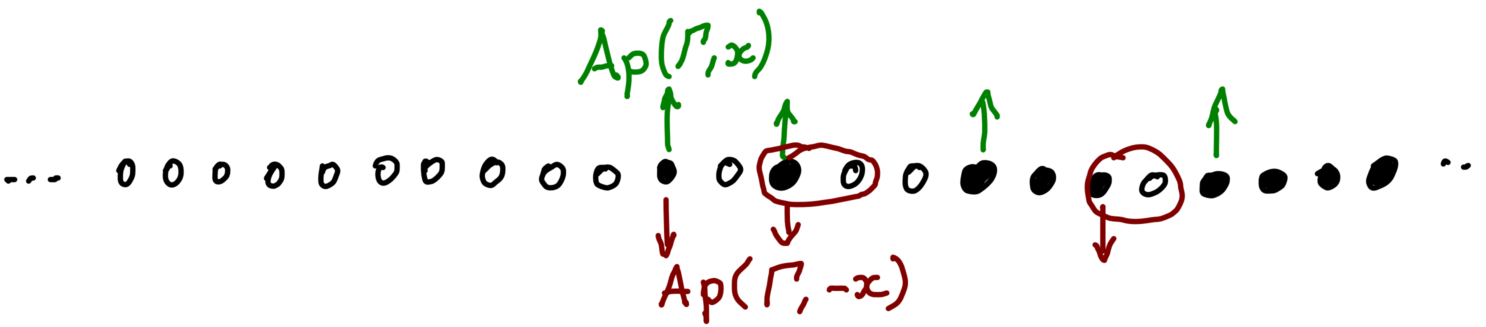

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$k_i - 1$ in $Ap(\Gamma, -x)$

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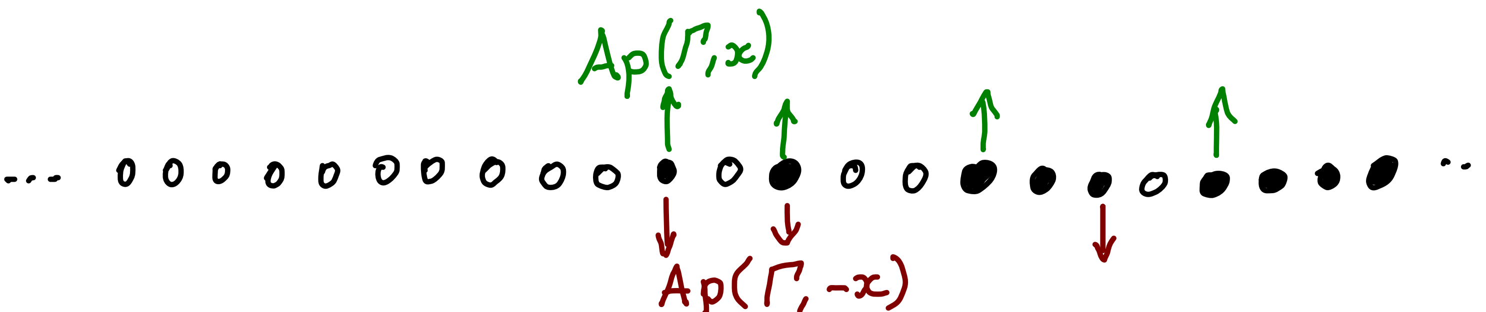

$$\sum_i k_i = \#Ap(\Gamma, x)$$

$$\sum_i k_i - x = \#Ap(\Gamma, -x)$$

A cocycle inside the Apéry

- Let $s \in \Gamma^*$, there is a bijection

$$\omega : \mathbb{Z}_s \longrightarrow \mathrm{Ap}(\Gamma, s)$$

such that $\omega(i) \equiv i \mod s$.

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or $$\omega(i) + \omega(j) = \omega(i+j) + s * h(i,j).$$

Examples

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- $\mathrm{Ap}(\langle 4,5,6\rangle, 4) = \{\underset{(0)}{0}, \underset{(1)}{5}, \underset{(2)}{6}, \underset{(3)}{11}\}$

$h(i,j)$	0	1	2	3
0				
1				
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$5 + 5 = 10 = 6 + 4 \cdot 1$

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- $\omega(i) = \sum_{j \in \mathbb{Z}_s} h(i,j)$

More properties

- It detects divisors

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- And maximal elements (and so Pseudo-Frobenius)

$$\max_{\leq_\Gamma} Ap(\Gamma, s) = \{\omega(i) \in Ap(\Gamma, s) \mid h(i,j) > 0,\ \forall j \in \mathbb{Z}_s \setminus \{0\}\}$$

Gluing of numerical semigroups

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- $a_2 \in \Gamma_1$, $a_1 \in \Gamma_2$, $\gcd(a_1, a_2) = 1$

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$$Ap(\Gamma, a_1 \cdot a_2) = a_1 \cdot Ap(\Gamma_1, a_2) + a_2 \cdot Ap(\Gamma_2, a_1)$$

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- $\Gamma = a_1 \Gamma_1 + a_2 \mathbb{N}$, $z = a_1 k + a_2 \omega$, $0 \leq \omega < a_1$

$$Ap(\Gamma, z) = \left(a_1 Ap(\Gamma_1, k + a_2) + a_2 \cdot \{0, \dots, \omega - 1\}\right)$$
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$$\# Ap(\Gamma, z) = \omega \cdot \# Ap(\Gamma_1, k+a_2) + (a_1 - \omega) \cdot \# Ap(\Gamma_1, k)$$

Gluing of numerical semigroups

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- $\Gamma = a_1 \mathbb{N} + a_2 \Gamma_2, \quad a_2 > a_1$

$$E(\Gamma, 2) = a_1 = m(\Gamma)$$

Free and telescopic

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  with $\Gamma_1 = \langle n_1, \dots, n_{e-1} \rangle$ telescopic and

  $$a_2 > a_1 \cdot n_{e-1}$$

Free and telescopic

<u>Theorem</u>: If Γ is telescopic

$$E(\Gamma, 2) = m(\Gamma)$$

Examples

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$$E(H_{q,r}, 2) = q^{r-1}$$

Examples

Generalized Suzuki numerical semigroups

$$S_{p,n} = \langle p^{2n+1}, p^{2n+1}+p^n, p^{2n+1}+p^{n+1}, p^{2n+1}+p^{n+1}+1 \rangle$$

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$$E(S_{p,n},2) = p^{2n+1} - p^{2n} + p^{n}$$

Thanks!