Comatrix Corings and Reconstruction*

J. Gómez-Torrecillas
Universidad de Granada

Mathematics Institute-Warwick, July 9th, 2004

*Based on the joint paper "Infinite comatrix corings" with **L. El Kaoutit**, IMRN 39 (2004), 2017-2037

Corings.(Following *Sweedler*, 1975) Let A be a ring. An A–*coring* is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \epsilon_{\mathfrak{C}})$ which consists of one A–bimodule $\mathfrak{C}$ and two homomorphism of A–bimodules

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_A \mathfrak{C} \qquad \epsilon_{\mathfrak{C}} : \mathfrak{C} \longrightarrow A \tag{1}$$

such that the diagrams

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ {\scriptstyle \Delta_{\mathfrak{C}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{C} \underset{A}{\otimes} \Delta_{\mathfrak{C}}} \\ \mathfrak{C} \otimes_A \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}} \underset{A}{\otimes} \mathfrak{C}} & \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mathfrak{C} \underset{A}{\otimes} \epsilon_{\mathfrak{C}}} \\ & & \mathfrak{C} \otimes_A A \end{array} \qquad \begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \epsilon_{\mathfrak{C}} \underset{A}{\otimes} \mathfrak{C}} \\ & & A \otimes_A \mathfrak{C} \end{array}$$

commute.

Example. *Sweedler's canonical coring*. Consider $B \leq A$ a subring.

Bimodule:

$$A \otimes_B A, \quad a(a' \otimes a'')a''' = aa' \otimes a''a'''$$

Comultiplication:

$$\Delta : A \otimes_B A \longrightarrow A \otimes_B A \otimes_A A \otimes_B A$$

$$a \otimes a' \longmapsto a \otimes 1 \otimes 1 \otimes a'$$

Counity:

$$\epsilon : A \otimes_B A \longrightarrow A, \quad a \otimes a' \longmapsto aa'$$

Example. *Idempotent coring.*

Bimodule: A twosided ideal I such that $I^2 = I$ and ${}_A A/I$ or A/I_A is flat.

Comultiplication: The canonical isomorphism $I \cong I \otimes_A I$.

Counity: The inclusion $I \subseteq A$.

Example. *Coring associated to a graded ring.* Consider $A = \bigoplus_{g \in G} A_g$ a ring graded by a group G.

Bimodule:

AG, the free *left* A–module with basis G

right action: $g * a_h = a_h gh$ for $a_h \in A_h$

Comultiplication:

$$\Delta : AG \longrightarrow AG \otimes_A AG$$

$$g \longmapsto g \otimes g$$

Counity:

$$\epsilon : AG \longrightarrow A, \quad g \longmapsto 1$$

More Examples. *Coring associated to a Hopf-comodule algebra and, more generally, to a entwining structure between an algebra and a coalgebra (Brzezinski-Takeuchi)*

Comodule categories. Given an A–coring $\mathfrak{C}$, the category $\mathcal{M}^{\mathfrak{C}}$ of all right $\mathfrak{C}$–comodules is defined as follows.

Objects: pairs (M, ρ_M), with M_A a module, and $\rho_M : M \to M \otimes_A \mathfrak{C}$ a morphism of A–modules such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \underset{A}{\otimes} \mathfrak{C} \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle M \underset{A}{\otimes} \Delta_{\mathfrak{C}}} \\ M \underset{A}{\otimes} \mathfrak{C} & \xrightarrow{\rho_M \underset{A}{\otimes} \mathfrak{C}} & M \underset{A}{\otimes} \mathfrak{C} \underset{A}{\otimes} \mathfrak{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho_M} & M \underset{A}{\otimes} \mathfrak{C} \\ & \searrow^{\cong} & \downarrow{\scriptstyle M \underset{A}{\otimes} \epsilon_{\mathfrak{C}}} \\ & & M \underset{A}{\otimes} A \end{array}$$

commute.

Morphisms: a morphism $f : (M, \rho_M) \to (N, \rho_N)$ is a morphism of A–modules $f : M \to N$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_N} \\ M \underset{A}{\otimes} \mathfrak{C} & \xrightarrow{f \underset{A}{\otimes} \mathfrak{C}} & N \underset{A}{\otimes} \mathfrak{C} \end{array}$$

$\mathcal{M}^{\mathfrak{C}}$ is an additive category with inductive limits, but it is not abelian in general (kernels can fail).

$$\begin{array}{c} \mathcal{M}^{\mathfrak{C}} \\ U \downarrow \quad \uparrow -\otimes_A \mathfrak{C} \\ \mathcal{M}_A \end{array}$$

the forgetful functor U has a right adjoint $-\otimes_A \mathfrak{C}$

Theorem. *The following are equivalent.*
(i) $\mathcal{M}^{\mathfrak{C}}$ is abelian and U is left exact;
(ii) $\mathcal{M}^{\mathfrak{C}}$ is a Grothendieck category and U is left exact;
(iii) ${}_A\mathfrak{C}$ is flat.

Remark. *$\mathcal{M}^{\mathfrak{C}}$ can be abelian without ${}_A\mathfrak{C}$ flat.*

Example. Let ${}_RB_S$ a bimodule, $A = \begin{pmatrix} R & B \\ 0 & S \end{pmatrix}$, and $I = I^2 = \begin{pmatrix} R & B \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{M}^I \sim \mathcal{M}_R$ but ${}_AI$ is no flat unless ${}_RB$ is.

Examples of categories of comodules

Descent data.

Coring: Sweedler's canonical coring $A \otimes_B A$ for a (commutative) ring extension $\psi : B \to A$.

Isomorphism of categories: $\mathcal{M}^{A\otimes_B A} \sim Desc(\psi)$ (a detailed proof in Caenepeel/Militaru/Zhu Springer LNM)

Graded modules.

Coring: Coring AG associated to a G–graded ring A.

Isomorphism of categories: $\mathcal{M}^{AG} \sim gr - A$.

Hopf modules.

Coring: Coring $A \otimes H$ associated to an H–comodule algebra A.

Isomorphism of categories: $\mathcal{M}^{A\otimes H} \sim \mathcal{M}_A^H$

Let $\mathfrak{C}$ be an A–coring with a group-like g,

$$T = \{a \in A | ag = ga\}$$

the subring of g–coinvariants of A, and

$$can : A \otimes_T A \to \mathfrak{C}$$

the canonical map ($a \otimes_T a' \mapsto aga'$). The following definition and theorem were given by T. Brzezinski, Alg. Repr. Theory, 2002.

Definition. *($\mathfrak{C}, g$) is Galois if* can *is a (coring) isomorphism.*

Theorem. *The following are equivalent.*
(i) ${}_A\mathfrak{C}$ is flat, and the functor

$$- \otimes_T A : \mathcal{M}_T \to \mathcal{M}^{\mathfrak{C}}$$

is an equivalence of categories;
(ii) $\mathfrak{C}$ is Galois and ${}_TA$ is faithfully flat.

Some consequences

1. For $\psi : B \to A$ commutative ring extension, with $\mathfrak{C} = A \otimes_B A$, and $g = 1 \otimes_B 1$, we have the faithfully flat descent: ψ is *effective* if and only if ${}_B A$ is faithfully flat.

2. For A a G–graded ring, with $\mathfrak{C} = AG$, and $g = e$, the neutral element of G, we easily deduce **Dade**'s characterization of *strongly graded rings*: $A - gr \sim \mathcal{M}_{A_e}$ if and only if $A_g A_h = A_{gh}$ for every $g, h \in G$.

3. For A an H–comodule algebra, with $\mathfrak{C} = A \otimes H$ and $g = 1 \otimes 1$, we have part of **Schneider**'s theorem: $\mathcal{M}_A^H \sim \mathcal{M}_{A^{coH}}$ if and only if $A^{coH} \subseteq A$ is H–Galois and ${}_{A^{coH}}A$ is faithfully flat.

Remarks: 1.- For a Galois coring with ${}_TA$ faithfully flat, A becomes a finitely generated projective generator for the category $\mathcal{M}^{\mathfrak{C}}$.
2.- The functor $- \otimes_T A$ is always left adjoint to the functor $\mathrm{Hom}_{\mathfrak{C}}(A, -)$, this last being isomorphic to the "coinvariants functor" defined by g.
3.- In favorable circumstances, the coring can be reconstructed from one of its representations (comodules).

The first two remarks are reminiscent of Mitchell's Theorem. Thus, a new question arises: for which corings $\mathfrak{C}$ has $\mathcal{M}^{\mathfrak{C}}$ a finitely generated projective generator? Could them be reconstructed from this comodule?

Remark: If $P \in \mathcal{M}^{\mathfrak{C}}$ is a small projective generator, then, by the adjunction $U \dashv - \otimes_A \mathfrak{C}$, P_A is small and projective, and, therefore, it is finitely generated and projective as a module.

A more general point of view

Let A be an algebra over a commutative ring K, and denote by add(A_A) the category of all finitely generated and projective right A–modules. Let

$$\omega : \mathcal{A} \to \mathsf{add}(A_A)$$

be a functor, where $\mathcal{A}$ is a K–linear small category. The situation where $\mathcal{A}$ is a subcategory of a Grothendieck category $\mathcal{C}$ is not rare (e.g., a category of comodules).

The idea is construct from the functor ω an A–coring $\mathfrak{R}(\omega)$ in such a way that the objects in $\mathcal{A}$ become right $\mathfrak{R}(\omega)$–comodules.

Let us start with the case where $\mathcal{A}$ has a single object, that is, it is a K–algebra B. The functor ω becomes a $B - A$–bimodule Σ such that $\Sigma_A \in$ add(A_A).

Comatrix Corings

Let ${}_B\Sigma_A$ be a $B-A$–bimodule; assume Σ_A is finitely generated and projective. Consider $\Sigma^* = \mathrm{Hom}_A(\Sigma, A_A)$ canonically as an $A-B$–bimodule. Pick $\{(e_i^*, e_i)\} \subseteq \Sigma^* \times \Sigma$ a dual basis.

Bimodule: $\Sigma^* \otimes_B \Sigma$, $a(\varphi \otimes u)a' = a\varphi \otimes ua'$.

Comultiplication:

$$\Sigma^* \otimes_B \Sigma \xrightarrow{\Delta} \Sigma^* \otimes_B \Sigma \otimes_A \Sigma^* \otimes_B \Sigma$$

$$\varphi \otimes_B u \longmapsto \sum_i \varphi \otimes_B e_i \otimes_A e_i^* \otimes_B u$$

Counity:

$$\Sigma^* \otimes_B \Sigma \xrightarrow{ev} A, \qquad \varphi \otimes_B u \longmapsto \varphi(u)$$

Then Σ_A becomes a right $\Sigma^* \otimes_B \Sigma$–comodule with the coaction

$$\varrho_\Sigma : \Sigma \to \Sigma \otimes_A \Sigma^* \otimes_B \Sigma \quad (u \mapsto \sum_i e_i \otimes_A e_i^* \otimes_B u)$$

Infinite comatrix corings

Returning to our general functor $\omega : \mathcal{A} \to \text{add}(A_A)$. For each $P \in \mathcal{A}$ consider the ring homomorphism

$$T_P = \text{End}_{\mathcal{A}}(P) \to S_P = \text{End}(P_A),$$

where we are denoting by P the image by ω in add(A_A) of $P \in \mathcal{A}$. Thus, every P becomes a $T_P - A$–bimodule with P_A finitely generated and projective. We have the coproduct of comatrix A–corings

$$\bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P$$

Every $P \in \mathcal{A}$ becomes a right comodule over this A–coring in an obvious way but this does NOT define in general a functor

$$\mathcal{A} \to \mathcal{M}^{\bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P}$$

.

In order to remedy this, we have to take the $T_Q - T_P$–bimodules

$$T_{PQ} = \mathsf{Hom}_{\mathcal{A}}(P, Q)$$

into account. In fact, T_{PQ} acts on the left on P (resp. on the right on Q^*) in a straightforward way. Using these actions, we have

Lemma. *The K–submodule $\mathfrak{J}(\omega)$ of*

$$\bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P$$

generated by the set

$$\{\varphi \otimes_{T_Q} tp - \varphi t \otimes_{T_P} p \;:\; \varphi \in Q^*, p \in P, t \in T_{PQ}, P, Q \in \mathcal{A}\}$$

is a coideal.

Proposition. *Define the factor A–coring*

$$\mathfrak{R}(\omega) = \frac{\bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P}{\mathfrak{J}(\omega)}$$

There is a functor $\mathfrak{F}(\omega) : \mathcal{A} \to \mathcal{M}^{\mathfrak{R}(\omega)}$ making the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \omega\ } & \mathrm{add}(A_A) \\ {\scriptstyle \mathfrak{F}(\omega)} \downarrow & & \downarrow \\ \mathcal{M}^{\mathfrak{R}(\omega)} & \xrightarrow[U]{} & \mathcal{M}_A \end{array}$$

commutative.

The right $\mathfrak{R}(\omega)$–comodule structure ϱ_P of $P \in \mathcal{A}$ is given explicitly as follows: choose a dual basis $\{(e^*_{\alpha_P}, e_{\alpha_P})\} \subseteq P^* \times P$, then

$$\varrho_P(u) = \sum_{\alpha_P} e_{\alpha_P} \otimes_A (e^*_{\alpha_P} \otimes_{T_P} u + \mathfrak{J}(\omega))$$

Now assume $\mathcal{A}$ to be a full subcategory of a K–linear category $\mathcal{C}$ such that the coproduct

$$\Sigma = \bigoplus_{P \in \mathcal{A}} P$$

does exist in $\mathcal{C}$, and assume further a functor $\Omega : \mathcal{C} \to \mathcal{M}_A$ which commutes with this coproduct and such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \omega\ } & \mathrm{add}(A_A) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\ \Omega\ } & \mathcal{M}_A \end{array}$$

Consider the ring $T = \mathrm{End}_{\mathcal{C}}(\Sigma)$; then Σ is a T–A–bimodule, and we have the $A-A$–bimodule $\Sigma^* \otimes_T \Sigma$.

Proposition. *There is a canonical surjective map of A–bimodules*

$$\Gamma : \bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P \to \Sigma^* \otimes_T \Sigma$$

whose kernel is just the coideal $\mathfrak{J}(\omega)$. Henceforth, there is a unique structure of A–coring on $\Sigma^* \otimes_T \Sigma$ such that Γ is a homomorphism of A–corings.

The map Γ has an explicit expression: for $P \in \mathcal{A}$ let

$$\iota_P : P \to \Sigma, \quad \pi_P : \Sigma \to P$$

be resp. the canonical injection and projection. Then

$$\bigoplus_{P\in\mathcal{A}} P^* \otimes_{T_P} P \xrightarrow{\quad\Gamma\quad} \Sigma^* \otimes_T \Sigma$$

$$(\varphi_P \otimes_{T_P} u_P)_{P\in\mathcal{A}} \longmapsto \sum_P \varphi_P \pi_P \otimes_T \iota_P(u_P)$$

and the comultiplication of $\Sigma^* \otimes_T \Sigma$ is given by

$$\Delta(\varphi \otimes_T x) = \sum_{P\in\mathcal{F}} \sum_{\alpha_P} \varphi \iota_P \pi_P \otimes_T \iota_P(e_{\alpha_P}) \otimes_A e^*_{\alpha_P} \pi_P \otimes_T \iota_P \pi_P(x),$$

where $\mathcal{F}$ is any finite set of objects of $\mathcal{A}$ such that $x = \sum_{P\in\mathcal{F}} \iota_P \pi_P(x)$. The counit of $\Sigma^* \otimes_T \Sigma$ is simply the evaluation map $\varphi \otimes_T x \mapsto \varphi(x)$.

We have thus the alternative description for the A–coring $\mathfrak{R}(\omega)$ as $\Sigma^* \otimes_T \Sigma$.

Now assume that $\mathcal{A}$ is a small subcategory of the category of right comodules $\mathcal{M}^{\mathfrak{C}}$ over an A–coring $\mathfrak{C}$, and that the functor $\omega : \mathcal{A} \to$ add(A_A) is the restriction of the forgetful functor $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$. We have then a pair of functors

$$\mathcal{M}_T \underset{\mathrm{Hom}_{\mathfrak{C}}(\Sigma,-)}{\overset{-\otimes_T \Sigma}{\rightleftarrows}} \mathcal{M}^{\mathfrak{C}}$$

where $- \otimes_T \Sigma$ is left adjoint to $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, -)$.

Now, using the counit of this adjunction and the isomorphism $\Sigma^* \cong \mathrm{Hom}_{\mathfrak{C}}(\Sigma, \mathfrak{C})$ we have

Lemma. *The map $can : \Sigma^* \otimes_T \Sigma \to \mathfrak{C}$ defined by $can(\varphi \otimes_T u) = (\varphi \otimes_A \mathfrak{C})\rho_{\Sigma}(u)$ is a homomorphism of A–corings.*

This map is called the *canonical map*. We will say that $(\mathfrak{C}, \mathcal{A})$ is *Galois* when can is an isomorphism.

As a consequence of Gabriel-Popescu Theorem, we have

Theorem. *(Reconstruction) Assume that* $\mathcal{M}^{\mathfrak{C}}$ *is abelian and it is generated by a set* $\mathcal{A}$ *of right comodules such that* $P_A \in \mathrm{add}(A_A)$ *for every* $P \in \mathcal{A}$*. Then* $(\mathfrak{C}, \mathcal{A})$ *is Galois.*

Example. *Let* C *be a coalgebra over a field, and* $\mathcal{A}$ *a generating set of finite dimensional right* C*–comodules. Then* can *gives an isomorphism of coalgebras*

$$C \cong \Sigma^* \otimes_T \Sigma$$

where $\Sigma = \bigoplus_{P \in \mathcal{A}} P$*.*

Example. *If an* A*–coring* $\mathfrak{C}$ *is cosemisimple (i.e.,* $\mathcal{M}^{\mathfrak{C}}$ *is a semisimple Grothendieck category), then*

$$\mathfrak{C} \cong \bigoplus_{P \in \mathcal{A}} P^* \otimes_{T_P} P$$

for $\mathcal{A}$ *a set of representatives of all simple right* $\mathfrak{C}$*–comodules. Hence, all* T_P *are here division rings.*

To see the connection of Galois comatrix corings with the faithfully flat descent, we shall give a third construction of (infinite) comatrix corings. Coming back to our category $\mathcal{C}$, consider the ring (without unit in general)

$$R = \bigoplus_{P \in \mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(Q, P),$$

which is a left ideal of $T = \mathrm{End}_{\mathcal{C}}(\Sigma)$. Then $\Sigma^{\dagger} = \bigoplus_{P \in \mathcal{A}} P^*$ becomes an $A - R$–bimodule.

Proposition. *We have a commutative diagram of surjective homomorphisms of A–bimodules.*

$$\begin{array}{ccc} \bigoplus_{P\in\mathcal{A}} P^* \otimes_{T_P} P & \xrightarrow{\Gamma_1} & \Sigma^{\dagger} \otimes_R \Sigma \\ \downarrow \Gamma & \swarrow \Gamma_2 & \\ \Sigma^* \otimes_T \Sigma & & \end{array}$$

Moreover, the kernel of Γ_1 is $\mathfrak{J}(\omega)$, and therefore $\Sigma^{\dagger} \otimes_R \Sigma$ is endowed with a structure of A–coring such that the former induces a commutative diagram of isomorphisms of A–corings

$$\begin{array}{ccc} \mathfrak{R}(\omega) & \xrightarrow{\simeq} & \Sigma^{\dagger} \otimes_R \Sigma \\ \downarrow \simeq & \swarrow \simeq & \\ \Sigma^* \otimes_T \Sigma & & \end{array}$$

Put $\mathcal{C} = \mathcal{M}^{\mathfrak{C}}$, for $\mathfrak{C}$ an A–coring, and let $\mathcal{A} = \{P_{\mathfrak{C}}\}$ be a set of comodules such that every P_A is finitely generated and projective.

We have a pair of functors

$$\mathcal{M}_A \underset{-\otimes_R \Sigma}{\overset{-\otimes_A \Sigma^\dagger}{\rightleftarrows}} \mathcal{M}_R$$

where $\mathcal{M}_A$ and $\mathcal{M}_R$ are categories of right unital modules. It is known that $-\otimes_R \Sigma$ is left adjoint to $-\otimes_A \Sigma^\dagger$. The counit of this adjunction is built from the evaluation map

$$ev : \Sigma^\dagger \otimes_R \Sigma \to A \quad (\varphi \otimes_R x \mapsto \sum \varphi_P(x_P)),$$

which is a homomorphism of A–bimodules.

Recall that for $N \in \mathcal{M}^{\mathfrak{C}}$ the cotensor product $N \Box_{\mathfrak{C}} \Sigma^\dagger$ is the equalizer

$$N \Box_{\mathfrak{C}} \Sigma^\dagger \xrightarrow{eq_{N,\Sigma^\dagger}} N \otimes_A \Sigma^\dagger \underset{N \otimes_A \lambda_{\Sigma^\dagger}}{\overset{\rho_N \otimes_A \Sigma^\dagger}{\rightrightarrows}} N \otimes_A \mathfrak{C} \otimes_A \Sigma^\dagger$$

This gives a functor $-\Box_{\mathfrak{C}} \Sigma^\dagger : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R$. Now, the adjunction isomorphism

$$\text{Hom}_A(M \otimes_R \Sigma, N) \cong \text{Hom}_R(M, N \otimes_A \Sigma^\dagger)$$

gives, by restriction, the isomorphism

$$\text{Hom}_{\mathfrak{C}}(M \otimes_R \Sigma, N) \cong \text{Hom}_R(M, N \Box_{\mathfrak{C}} \Sigma^\dagger)$$

which shows that in the pair of functors

$$\mathcal{M}^{\mathfrak{C}} \underset{-\otimes_R \Sigma}{\overset{-\Box_{\mathfrak{C}} \Sigma^\dagger}{\rightleftarrows}} \mathcal{M}_R$$

$- \otimes_R \Sigma$ is left adjoint to $-\Box_{\mathfrak{C}} \Sigma^\dagger$. The counit of this adjunction at $N \in \mathcal{M}^{\mathfrak{C}}$ is given by

$$\begin{array}{ccc} & N \otimes_A \Sigma^\dagger \otimes_R \Sigma & \\ {\scriptstyle eq_{N,\Sigma^\dagger} \otimes_R \Sigma} \nearrow & & \searrow {\scriptstyle N \otimes_A ev} \\ (N \Box_{\mathfrak{C}} \Sigma^\dagger) \otimes_R \Sigma & \xrightarrow{\qquad \delta_N \qquad} & N \end{array}$$

The Galois comodule structure Theorem, I

With the collaboration of J. Vercruysse (work in progress), or, alternatively, with the help of a result by Abrams and Menini, J.P.A.A., 1996, we have

Theorem. *If P_A is finitely generated and projective for every $P \in \mathcal{A}$, the following are equivalent.*
(i) $(\mathfrak{C}, \mathcal{A})$ is Galois and ${}_R\Sigma$ is flat;
(ii) ${}_A\mathfrak{C}$ is flat and $\mathcal{A}$ is a generating set of small (or f.g.) objects for $\mathcal{M}^{\mathfrak{C}}$;
(iii) ${}_A\mathfrak{C}$ is flat and δ_N is an isomorphism for every $N \in \mathcal{M}^{\mathfrak{C}}$.

With $\mathcal{A}$ a singleton, this has been formulated in Brzezinski-Wisbauer's corings book, (2003).

The Galois Comodule Structure, II

With the collaboration of L. El Kaoutit, and some help from Freyd/Gabriel's Theorem, we have (Int. Math. Res. Notices, 2004).

Theorem. *Let $\mathcal{A}$ be a set of right $\mathfrak{C}$–comodules. Consider the ring extension $R \subseteq S$, where $R = \oplus_{P,Q\in\mathcal{A}}\mathrm{Hom}_{\mathfrak{C}}(Q,P)$ and $S = \oplus_{P,Q\in\mathcal{A}}\mathrm{Hom}_{A}(Q,P)$. The following statements are equivalent.*
(i) P_A is f.g. projective for all $P \in \mathcal{A}$, $(\mathfrak{C},\mathcal{A})$ is Galois, and ${}_R\Sigma$ is faithfully flat;
(ii) ${}_A\mathfrak{C}$ is flat and $-\otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\mathfrak{C}}$ is an equivalence of categories;
(iii) ${}_A\mathfrak{C}$ is flat and $\mathcal{A}$ is a generating set of small projectives for $\mathcal{M}^{\mathfrak{C}}$;
(iv) ${}_A\mathfrak{C}$ is flat, P_A is f.g. projective for all $P \in \mathcal{A}$, $(\mathfrak{C},\mathcal{A})$ is Galois, and ${}_RS$ is faithfully flat.