Somewhat semi-commutative polynomials

(Talk)

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Let n be a positive integer. We consider $\mathbb{N}^n$ as additive monoid with the sum defined componentwise. Let $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n$ be the basis of this free abelian monoid.

Definition 1 An *admissible order* $\preceq$ on $(\mathbb{N}^n, +)$ is a total order such that
(a) $\mathbf{0} = (0, \ldots, 0) \preceq \boldsymbol{\alpha}$ for every $\boldsymbol{\alpha} \in \mathbb{N}^n$.
(b)$\boldsymbol{\alpha} + \boldsymbol{\gamma} \preceq \boldsymbol{\beta} + \boldsymbol{\gamma}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n$ with $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$.

By Dickson's Lemma, these are good orders (i.e., any non empty subset of $\mathbb{N}^n$ has a first element).

By $\preceq_{lex}$ we denote the *lexicographical order* with

$$\boldsymbol{\epsilon}_1 \preceq_{lex} \cdots \preceq_{lex} \boldsymbol{\epsilon}_n.$$

Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n), \boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$, define $|\alpha|_{\boldsymbol{w}} = \alpha_1 w_1 + \cdots + \alpha_n w_n$. The $\boldsymbol{w}$*-weighted admissible order* $\preceq_{\boldsymbol{w}}$ is defined by

$$\boldsymbol{\alpha} \preceq_{\boldsymbol{w}} \boldsymbol{\beta} \iff \begin{cases} |\boldsymbol{\alpha}|_{\boldsymbol{w}} < |\boldsymbol{\beta}|_{\boldsymbol{w}} \\ \text{or} \\ |\boldsymbol{\alpha}|_{\boldsymbol{w}} = |\boldsymbol{\beta}|_{\boldsymbol{w}} \text{ and } \boldsymbol{\alpha} \preceq_{lex} \boldsymbol{\beta}. \end{cases}$$

The *degree lexicographical order* $\preceq_{deglex}$ is obtained as $\preceq_{\boldsymbol{w}}$ with $\boldsymbol{w} = (1, \ldots, 1)$.

Let R be an algebra over a field $\mathbf{k}$, and let $x_1, \ldots, x_n$ be elements in R. A *standard monomial in* $x_1, \ldots, x_n$ is an expression $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Assume that an element $r \in R$ can be written in the form

$$r = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} r_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \qquad (r_{\boldsymbol{\alpha}} \in \mathbf{k}) \tag{1}$$

The expression (1) is called a *standard representation* of r. We will often refer as *polynomials* to the elements of R having a standard representation. Define

$$\mathcal{N}(r) = \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid r_{\boldsymbol{\alpha}} \neq 0\}$$

Definition 2 Let $\preceq$ be an admissible order on $\mathbb{N}^n$, and consider an element r of R having a standard representation (1). The *exponent* $\exp(r)$ of (the standard representation of) r is defined as

$$\exp(r) = \max_{\preceq} \mathcal{N}(r)$$

We stress that $\exp(r)$ depends on the given order $\preceq$ and on the standard representation (1) of r.

Definition 3 An algebra R over a field $\mathbf{k}$ is said to be a *Poincaré-Birkhoff-Witt* algebra if R is generated by finitely many elements $x_1, \ldots, x_n$ such that

(PBW1) The standard monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} \in \mathbb{N}^n$ form a basis of R as a $\mathbf{k}$–vector space.

(PBW2) There exists an admissible order $\preceq$ on $\mathbb{N}^n$ such that for every i, j with $1 \leqslant i < j \leqslant s$ there are non-zero scalars $q_{ji} \in \mathbf{k}$ and polynomials $p_{ji} \in R$ such that

$$x_j x_i = q_{ji} x_i x_j + p_{ji} \text{ with } \exp(p_{ji}) \prec \boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_j$$

Notice that R can be then thought as the algebra generated by $x_1, \ldots, x_n$ subjected to the relations

$$Q \equiv x_j x_i = q_{ji} x_i x_j + p_{ji},\ 1 \leqslant i < j \leqslant n$$

We will then say that $R = \mathbf{k}\{x_1, \ldots, x_n; Q, \preceq\}$ is a PBW algebra.

Here, some pertinent comments about Definition 3 arise.

- The notion of PWB algebra is a restatement of the concept of *solvable polynomial algebra* introduced by **Kandri-Rody** and **Weispfenning** in 1990. So, all fundamental algorithmic tools (Gröbner bases...) are available here. In fact, these algorithms are implemented in **Kredel**'s `MAS Modula-2 Algebra System`.

- The more general concept of *PBW ring* over a skew field (**Bueso, G-T, Lobillo**, 1998), embodies **Kredel**'s *solvable polynomial rings* (1992).

- The condition (PBW2) is equivalent (under condition (PBW1)) to

$$\exp(fg) = \exp(f) + \exp(g) \tag{2}$$

for every $f, g \in R$.

- If $R = \mathbf{k}\{x_1, \ldots, x_n; Q, \preceq\}$ is a PBW algebra and $\preceq'$ is a different admissible order on $\mathbb{N}^n$, then we cannot expect in general that R is a PBW with respect to the new order $\preceq'$.

Assume R is generated by $x_1, \ldots, x_n$ and suppose we know that these generators satisfy some *somewhat semi-commuting relations*

$$Q \equiv x_j x_i = q_{ji} x_i x_j + p_{ji} \qquad (1 \leqslant i < j \leqslant n)$$

where $0 \neq q_{ji} \in \mathbf{k}$ and $p_{ji} \in R$ are polynomials.

Question: Is R a PBW algebra? This leads to

Question 1: Are the monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ a $\mathbf{k}$–basis of R?

Question 2: Are the relations *bounded* in the sense that there is and admissible order $\preceq$ on $\mathbb{N}^n$ such that

$$\exp(p_{ji}) \prec \boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_j$$

for every $i < j$?

Answer to the **Question 2:** Given the relations

$$\boxed{Q \equiv x_j x_i = q_{ji} x_i x_j + p_{ji}}$$

write $A_{ji} = \mathcal{N}(p_{ji})$ and

$$C = \bigcup_{1 \leqslant i < j \leqslant n} C_{ji},$$

where $C_{ji} = \boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_j - A_{ji} \subseteq \mathbb{Z}^n$

Proposition 4 *The relations* Q *are bounded for some admissible order on* $\mathbb{N}^n$ *if and only if the following linear programming problem has a solution.*

$$\begin{array}{l} \textbf{\textit{minimize}}\, f(\boldsymbol{w}) = w_1 + \cdots + w_n \\ \textbf{\textit{with the constraints}} \\ \Phi \equiv \begin{cases} w_i \geqslant 1 & (i = 1, \dots n) \\ |\boldsymbol{\alpha}|_{\boldsymbol{w}} \geqslant 1 & (\boldsymbol{\alpha} \in C) \end{cases} \end{array} \tag{3}$$

In such a case, for any vector $\boldsymbol{w} \in \Phi \cap \mathbb{N}^n_+$ *we have* $|\exp(p_{ji})|_{\boldsymbol{w}} < w_i + w_j$

$$(\textit{and, hence, } \exp(p_{ji}) \prec_{\boldsymbol{w}} \boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_j)$$

for every i, j.

Note: The preceding proposition can be deduced from basic facts on convex polytopes (**Caratheodory**'s theorem), or from the papers by **Mora and Robbiano** (1988) and **Weispfenning** (1987) on universal Gröbner bases.

Corollary 5 *If* $R = \mathbf{k}\{x_1, \ldots, x_n; Q, \preceq\}$ *is a PBW algebra then we can compute vectors* $\boldsymbol{w} \in \mathbb{N}_+^n$ *such that* $R = \mathbf{k}\{x_1, \ldots, x_n; Q, \preceq_{\boldsymbol{w}}\}$ *is a PBW algebra.*

As a consequence of this corollary, we can give an algorithm for the computation of the Gelfand-Kirillov dimension for finitely generated modules over any PBW algebra (**Bueso, G-T, Lobillo** (1998)).

Example 6

$$\left\{\begin{array}{lcl} yx & = & qxy + z \\ zx & = & qxz + x^3 \\ yz & = & qzy + y^2 \end{array}\right. \qquad (4)$$

The associated linear programming problem is

minimize $f(\mathrm{w}) = w_x + w_y + w_z$
with the constraints
$w_x \geq 1$
$w_y \geq 1$
$w_z \geqslant 1$
$w_x + w_y - w_z \geqslant 1$
$-2w_x + w_z \geqslant 1$
$-w_y + w_z \geqslant 1$

The minimum is situated at $\left\{w_x = 2, w_z = 5, w_y = 4\right\}$, so we may conclude that the relations (4) are $\preceq_{\boldsymbol{w}}$-bounded for the vector $\boldsymbol{w} = (2, 4, 5)$.

minimize:

$$f_{12} + f_{13} + f_{23} + k_1 + k_2 + l_1 + l_2 + e_{12} + e_{13} + e_{23}$$

with the constraints:

$$f_{12} \geqslant 1,\ f_{13} \geqslant 1,\ f_{23} \geqslant 1$$
$$k_1 \geqslant 1,\ k_2 \geqslant 1$$
$$l_1 \geqslant 1,\ l_2 \geqslant 1$$
$$e_{12} \geqslant 1,\ e_{13} \geqslant 1,\ e_{23} \geqslant 1$$
$$f_{12} - f_{13} + f_{23} \geqslant 1$$
$$e_{12} - e_{13} + e_{23} \geqslant 1$$
$$-2k_1 - f_{23} + e_{12} + f_{13} \geqslant 1$$
$$-2k_2 - e_{12} + e_{13} + f_{23} \geqslant 1$$
$$-2l_1 - e_{23} + e_{13} + f_{12} \geqslant 1$$
$$-f_{12} - 2l_2 + e_{23} + f_{13} \geqslant 1$$
$$-2k_1 + e_{12} + f_{12} \geqslant 1$$
$$-2l_1 + e_{12} + f_{12} \geqslant 1$$
$$-2k_1 - 2k_2 + e_{13} + f_{13} \geqslant 1$$
$$-2l_1 - 2l_2 + e_{13} + f_{13} \geqslant 1$$
$$-2k_2 + e_{23} + f_{23} \geqslant 1$$
$$-2l_2 + e_{23} + f_{23} \geqslant 1$$

Minimum is at:

$$f_{23} = 3, e_{13} = 1, k_1 = 1, e_{12} = 1, e_{23} = 1, l_2 = 1,$$
$$f_{12} = 3, l_1 = 1, f_{13} = 5, k_2 = 1$$

Definition 7 (**McConnell**). A graded $\mathbf{k}$-algebra $A = \oplus_{s\geqslant 0} A_s$ is called *semi-commutative* if R is generated as $\mathbf{k}$–algebra by a finite set of homogeneous elements $y_1, \ldots, y_n$ such that there are non zero scalars $q_{ji} \in \mathbf{k}$ such that $y_j y_i = q_{ji} y_i y_j$, $1 \leqslant i < j \leqslant n$.

Clearly, the standard monomials $\boldsymbol{y}^{\boldsymbol{\alpha}} = y_1^{\alpha_1} \ldots y_n^{\alpha_n}$ with $\boldsymbol{\alpha} \in \mathbb{N}^n$ span A as a $\mathbf{k}$–vector space. Therefore, every element of A has at least a standard representation.

Definition 8 A semi-commutative graded algebra is said to be a *graded quantum affine space* if the monomials $\boldsymbol{y}^{\boldsymbol{\alpha}}$ are $\mathbf{k}$–linearly independent (and hence, they form a $\mathbf{k}$–basis for A).

Of course, every graded quantum affine space is a PBW algebra with respect to any admissible order. Our aim is to show that the PBW algebras are, precisely, those filtered algebras which have a quantum affine space as associated graded algebra.

An algebra R over a field $\mathbb{k}$ is said to be a *filtered* $\mathbb{k}$-algebra, if it is endowed with an ascending chain $FR = \{F_s R;\ s \geqslant 0\}$ of vector subspaces, the "filtration" of R, satisfying for all $s, t \geqslant 0$

1. $1 \in F_0 R$;
2. $F_s R \subseteq F_{s+1} R$;
3. $(F_s R)(F_t R) \subseteq F_{s+t} R$;
4. $R = \bigcup_{s \geqslant 0} F_s R$.

The *associated graded algebra* is defined as the $\mathbf{k}$–vector space

$$\mathrm{gr}(R) = \bigoplus_{s \geqslant 0} \frac{F_s R}{F_{s-1} R}$$

endowed with the product defined on homogeneous elements as

$$(f + F_{s-1} R)(g + F_{t-1} R) = fg + F_{s+t-1} R$$

Proposition 9 *Let R be a filtered $\mathbf{k}$-algebra with filtration FR such that $\mathrm{gr}(R)$ is semi-commutative generated by homogeneous elements $y_1, \ldots, y_n$ with $\deg(y_i) = u_i \geqslant 0$ for $1 \leqslant i \leqslant n$. Put $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ and, for $1 \leqslant i < j \leqslant n$, let $0 \neq q_{ji} \in \mathbf{k}$ such that $y_j y_i = q_{ji} y_i y_j$.*

1. *If $x_1, \ldots, x_n \in R$ are such that $x_i = y_i + F_{u_i - 1}R$ for $1 \leqslant i \leqslant n$ then*

$$F_s R = \sum_{|\alpha|_{\boldsymbol{u}} \leqslant s} \mathbf{k}\boldsymbol{x}^{\boldsymbol{\alpha}}$$

   and, therefore, R satisfies a set Q of $\preceq_{\boldsymbol{u}}$-bounded relations

$$Q \equiv x_j x_i = q_{ji} x_i x_j + p_{ji} \ \ \text{with} \ \ \exp(p_{ji}) \prec_{\boldsymbol{u}} \boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_j$$

   for some polynomials $p_{ji} \in R$.

2. *If $\mathrm{gr}(R)$ is a graded quantum affine space, then $R = \mathbf{k}\{x_1, \ldots, x_n; Q, \preceq_{\boldsymbol{u}}\}$ is a PBW algebra.*

Proposition 10 *Let R be a $\mathbf{k}$-algebra generated by finitely many elements $x_1, \ldots, x_n$ and assume that there is a vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$, such that R satisfies a set Q of $\preceq_{\boldsymbol{w}}$-bounded relations*

$$Q \equiv x_j x_i = x_i x_j + p_{ji} \qquad \textit{with } \exp(p_{ij}) \prec_{\boldsymbol{w}} \boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_j$$

Define, for every positive integer s,

$$F_s^{\boldsymbol{w}} R = \sum_{|\alpha|_{\boldsymbol{w}} \leq s} \mathbf{k} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Then

1. *The algebra R is filtered with filtration $F^{\boldsymbol{w}} R = \{F_s^{\boldsymbol{w}} R\}$.*

2. *The associated graded algebra $\mathrm{gr}^{\boldsymbol{w}}(R)$ is generated by the elements $y_i = x_i + F_{w_i - 1}^{\boldsymbol{w}} R$ for $1 \leqslant i \leqslant n$.*

3. *If $h_{ji} = p_{ji} + F_{w_i + w_j - 1}^{\boldsymbol{w}} R$ then the following relations are satisfied in $\mathrm{gr}^{\boldsymbol{w}}(R)$ for $1 \leqslant i < j \leqslant n$*

$$Q_w \equiv y_j y_i = q_{ji} y_i y_j + h_{ji} \quad \textit{with } \exp(h_{ij}) \prec_{\boldsymbol{w}} \boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_j$$

4. *If $R = \mathbf{k}\{x_1, \ldots, x_n; Q, \preceq_{\boldsymbol{w}}\}$ is a PBW algebra, then $\mathrm{gr}^{\boldsymbol{w}}(R) = \mathbf{k}\{y_1, \ldots, y_n; Q_w, \preceq_{\boldsymbol{w}}\}$ is a PBW algebra.*

Theorem 11 *The following conditions are equivalent for a* $\mathbf{k}$*–algebra* R*.*

(i) R *is filtered by a finite-dimensional filtration with semi-commutative associated graded algebra.*

(ii) R *is filtered with semi-commutative associated graded algebra.*

(iii) R *satisfies a set* Q *of* $\preceq$*–bounded somewhat semi-commuting relations for some admissible order* $\preceq$*.*

(iv) R *satisfies a set* Q *of* $\preceq_{\boldsymbol{w}}$*–bounded somewhat semi-commuting relations for some vector* $\boldsymbol{w} \in \mathbb{N}_+^n$*.*

Theorem 12 *The following conditions are equivalent for a* $\mathbf{k}$*–algebra* R.

(i) R *is filtered by a finite-dimensional filtration with* $\mathrm{gr}(R)$ *a graded affine quantum space.*

(ii) R *is filtered with* $\mathrm{gr}(R)$ *a graded affine quantum space.*

(iii) R *is a PBW algebra.*

(iv) R *is a PBW algebra with respect to some admissible order of the form* $\preceq_{\boldsymbol{w}}$ *with* $\boldsymbol{w} \in \mathbb{N}_{+}^{n}$